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THESIS

CONTROLLABILITY OF NON-NEWTONIAN FLUIDS UNDER HOMOGENEOUS FLOWS

by

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September 2007

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The ability to control a viscoelastic field is an essential concept that defines some important restrictions and potentials of the influenced material. This thesis investigates the controllability of three popular constitutive models under homogeneous extensional and shear flows via the Lie bracket method. The constitutive models are as follows: the Phan-Thien-Tanner model; the Johnson-Segalman model; and the Doi model. The effect of extensional flow on these models and the effect of shear flow on the Doi model have not been explored previous to this work. The main contribution of this thesis is to characterize the submanifolds in the state space on which the non-Newtonian flow fields are weakly controllable. This kind of approach based on the control Lie algebra can be applied to a wider variety of complex models.

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CONTROLLABILITY OF NON-NEWTONIAN FLUIDS UNDER HOMOGENEOUS FLOWS

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ABSTRACT

The ability to control a viscoelastic field is an essential concept that defines some important restrictions and potentials of the influenced material. This thesis investigates the controllability of three popular constitutive models under homogeneous extensional and shear flows via the Lie bracket method. The constitutive models are as follows: the Phan-Thien-Tanner model; the Johnson-Segalman model; and the Doi model. The effect of extensional flow on these models and the effect of shear flow on the Doi model have not been explored previous to this work. The main contribution of this thesis is to characterize the submanifolds in the state space on which the non-Newtonian flow fields are weakly controllable. This kind of approach based on the control Lie algebra can be applied to a wider variety of complex models.

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I. INTRODUCTION

A. BACKGROUND AND PURPOSE

Controllability of viscoelastic fluids is a characteristic that is significant in the design of desired materials. Though the theory of control for linear systems has been aptly developed, nonlinear problems remain unfamiliar. Recent studies from Renardy [1] investigated the nonlinear case and characterized controllability and the associated reachable set of several popular constitutive models under homogeneous shear flow of viscoelastic fluids. Analysis revealed that for some of the models the states in stress space are specified by a positive definiteness inequality of the stress tensor and are reachable from a given initial condition. Renardy followed up with additional research which extended to the control of nonhomogeneous shear flow, explicitly to the upper convected Maxwell fluid [2].

The purpose of this thesis is to broaden Renardy's research [1] on the controllability of viscoelastic fluids by incorporating a variety of homogeneous steady flow fields. This investigation encompasses nonlinear characteristics, thus the method of analysis presented is principally based upon the nonlinear geometric control theory, which differs from Renardy's work [1]. An additional difference is the definition of controllability, where this paper focuses on a local adaptation of the concept, known as weak controllability. The definition of weak controllability has been extensively used in nonlinear control theory and engineering applications. In practice engineers prefer to attain a distance target by making a sequence of local movements to reduce the side effects of model uncertainties, system perturbations and sensor noises. So, weak controllability provides more practical information for engineering applications.

Though the study here can be applied to other complicated constitutive models, this thesis will focus on the following three representative fluids: the Phan-Thien-Tanner; the Johnson-Segalman; and the Doi. The first two are popular models for viscoelastic fluids whereas the last one is a classical model for liquid crystalline polymers.

B. CONTROLLABILITY OF CONTROL SYSTEMS

For reader's convenience, we present some basic definitions and theorems adopted from [4]. More details can be found in [3].

Let

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \tag{1}$$

be a general nonlinear control system, described to be affine in control where x is the state variable, and the control variables are $u_i \in \mathbb{R}$ for $i = 1, \ldots, m$. Overall, the state variable obtains a value on a n-dimensional smooth manifold denoted by M. For systems described by (1), *controllability* is a significant property, and plays a crucial role in many control problems. Controllability distinguishes the manipulation capability of the system under control, and the stabilization of unstable systems via feedback. There exists a substantial amount of literature on controllability theory. For this thesis, the geometric approach from Isidori [3] will be adopted. Definitions are as follows:

Definition 1 [4]. *A point x_1 in M is characterized as reachable from x_0 if there exists piecewise constant input functions, $u_1 = \alpha_i(t)$, such that the corresponding trajectory $x(t)$ with initial state $x(0) = x_0$ reaches $x(T) = x_1$ in finite time for some* $0 < T$.

For nonlinear control systems, the global reachability is usually very difficult to prove. Instead, a practical solution is to study weak controllability.

Definition 2 [4]. *A control system is characterized as weakly controllable within some open subset $S \subseteq M$, if for each x_0* in *S, there is an open region U_0 of x_0 such that the set of points reachable from x_0 to $x(t)$ within U_0 encloses at least one open subset of M*.

A weakly controllable system indicates that the locally reachable states create a "solid", non empty region. The theory of controllability is more restrictive than weak

controllability such that it mandates any two points in M must be reachable from each other. In linear control systems, concepts of controllability and weak controllability are interchangeable. Alternatively, with nonlinear control systems (such as the ones presented in this thesis), characterizing the controllability of the system would require information on the global geometric properties of the vector fields. For general control systems, this remains to be a challenging quandary. Interestingly enough, by exploiting the dimension of the control Lie algebra and a distribution produced by the vector fields related to the control system, the weak controllability can be characterized. This brings to light several significant properties of a control system [4].

Consider the notion of smooth vector fields defined on some manifold M. These vectors, such as $f(x)$ and $g_i(x)$ of (1), can be given different algebraic structures which can be useful in understanding the controllability of the system. Lie Algebra, for instance is such a structure in which the product of vector fields is defined by their Lie bracket, $[f,g]$. It is within this operation where the minimum subalgebra containing $f, g_1, \ldots, g_m$ is described as the *Control Lie Algebra*, symbolized by $\mathcal{C}$. Thus, within manifold M, at every point x, the vectors in $\mathcal{C}$ span a vector space. These vectors produce a distribution defined by $\Delta_{\mathcal{C}}(x) = span\{X(x) \mid X \text{ is a vector field in } \mathcal{C}\}$.

Definition 3 [4]. Let n *be the dimension of manifold* M. *A control system satisfies the controllability rank condition (CRC) on an open set* $S \subset M$ *if* $\dim(\Delta_{\mathcal{C}}(x)) \equiv n$ *for all* x *in* S.

Theorem 1 [3]. Let n *be the dimension of manifold* M. *A sufficient condition for control system (1) to be weakly controllable on an open set* S *is for the* $\dim(\Delta_{\mathcal{C}}(x)) \equiv n$ *for all* x *in* S. Or said differently, the control system (1) is weakly controllable if it satisfies the controllability rank condition on S.

C. FLOW FIELDS

Figure 1 and Figure 2 are graphs of the flow fields that will be used in this thesis. Figure 1 represents a *homogeneous extensional flow* where the elongational rate is

constant and the velocity is described as: $\boldsymbol{v} = \left(\dot{\gamma}(t)\frac{x}{2}, -\dot{\gamma}(t)\frac{y}{2} \right)$. Upon investigating Figure 1, one can see that there is a vertical contraction and a horizontal extension.

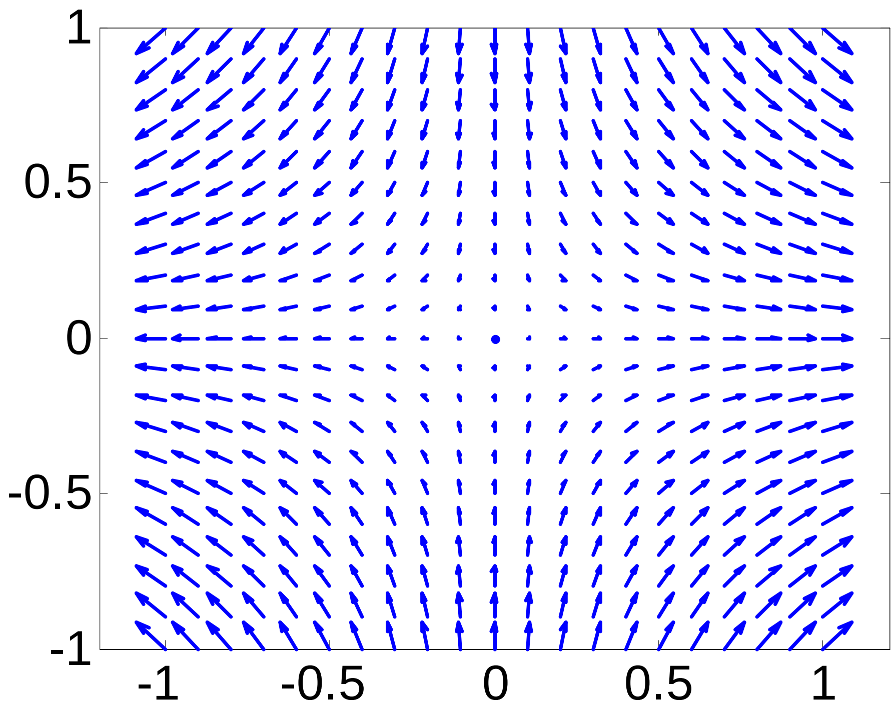

Figure 1. An Example of An Extensional Flow

Figure 2 represents a *homogeneous shear flow* where shear rate is constant and the velocity is described as: $\boldsymbol{v} = \left(\dot{\gamma}(t)\, y, 0 \right)$. In investigating Figure 2, one can see that there is a shearing motion in the horizontal direction.

Shear flow and extensional flow have many physical applications. For example, in extrusion manufacturing the flow away from boundaries is well approximated locally by elongation; in film and sheet manufacturing, or in squeezing flow between parallel disks, there are interior flow regions approximated by elongation. In fact, many manufacturing process flows can be decomposed as a combination of shear flow and extensional flow.

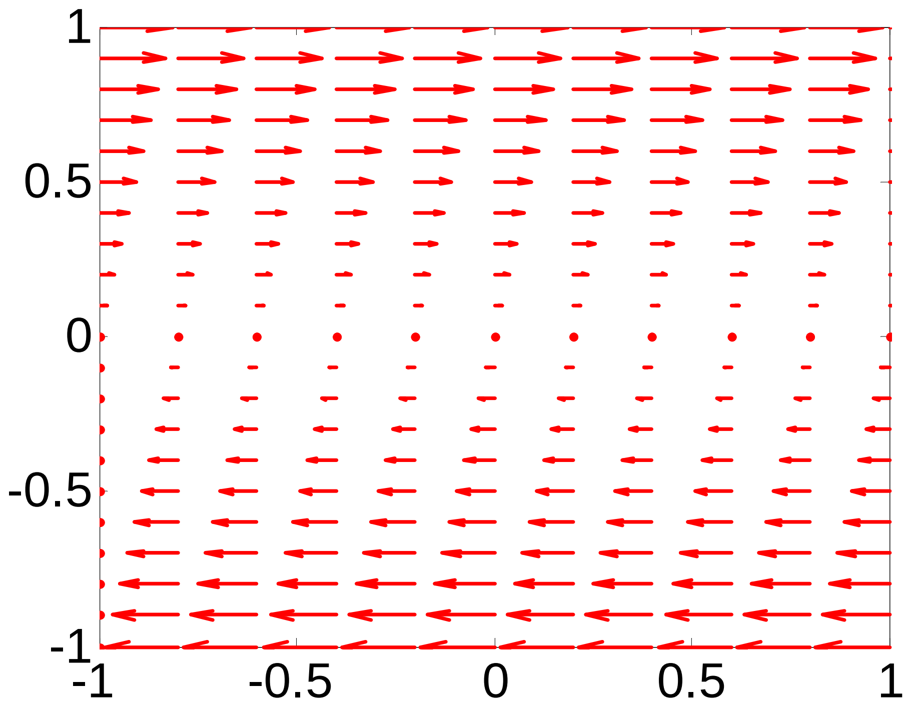

Figure 2. An Example of A Shear Flow

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II. THE PHAN-THIEN-TANNER MODEL

Of the nonlinear viscoelastic constitutive equations, one of the most applied differential type is the Phan-Thien-Tanner model [5]. The Phan-Thien-Tanner model has two parameters that control non-linearity and has the ability to fit, to some extent, the shear and elongational properties independently. It gives, however, spurious oscillations during start-up of shear flow. Thus the model is still liable for improvement.

The Phan-Thien-Tanner model is described by the following state equation:

$$\dot{\boldsymbol{T}} - (\nabla \boldsymbol{v})\boldsymbol{T} - \boldsymbol{T}(\nabla \boldsymbol{v})^T + \lambda \boldsymbol{T} + \kappa(tr\boldsymbol{T})\boldsymbol{T} = 2\mu \boldsymbol{D} \tag{2}$$

where the definition of its element are as follows:

$\boldsymbol{T}$: Stress tensor $\Rightarrow \boldsymbol{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ (where $T_{12} = T_{21}$ due to symmetry)

$\dot{\boldsymbol{T}}$: Derivative of the stress tensor with respect to time

$\boldsymbol{v}$: Velocity where $\boldsymbol{v} = (v_1, v_2)$

$\nabla \boldsymbol{v}$: Gradient of the velocity $\Rightarrow \nabla \boldsymbol{v} = \begin{bmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{bmatrix}$

$(\nabla \boldsymbol{v})^T$: Gradient of the velocity transposed

λ : Relaxation rate

κ : Transposed velocity gradient tensor

$tr\boldsymbol{T}$: Trace of the stress tensor

μ : Elastic modulus

$\boldsymbol{D}$: Rate of deformation tensor (the symmetric part of $\nabla \boldsymbol{v}$) $\Rightarrow \boldsymbol{D} = \frac{\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T}{2}$

$\dot{\gamma}$: Control input (closely related to velocity)

A. HOMOGENOUS EXTENSIONAL FLOW

A fluid in a homogeneous extensional flow with rate $\dot{\gamma}(t)$ is defined by velocity $\boldsymbol{v} = \left(\dot{\gamma}(t)\frac{x}{2}, -\dot{\gamma}(t)\frac{y}{2}\right)$, where the velocity gradient tensor is $\nabla\boldsymbol{v} = \frac{\dot{\gamma}(t)}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$. Recall, a schematic plot of extensional flow is given in Figure 1 of section I. This velocity is applied to the system (2), whereby the system's components are investigated.

$$\Rightarrow \dot{\boldsymbol{T}} + \frac{-\dot{\gamma}(t)}{2}\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\boldsymbol{T} + \boldsymbol{T}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right) + \left(\lambda + \kappa\left(tr\boldsymbol{T}\right)\right)\boldsymbol{T} = \mu\dot{\gamma}(t)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$\Rightarrow \dot{\boldsymbol{T}} + \frac{-\dot{\gamma}(t)}{2}\left(\begin{bmatrix}T_{11} & T_{12}\\ -T_{12} & -T_{22}\end{bmatrix} + \begin{bmatrix}T_{11} & -T_{12}\\ T_{12} & -T_{22}\end{bmatrix}\right) + \left(\lambda + \kappa\left(tr\boldsymbol{T}\right)\right)\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix} = \mu\dot{\gamma}(t)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$\Rightarrow \dot{\boldsymbol{T}} + \frac{-\dot{\gamma}(t)}{2}\begin{bmatrix}2T_{11} & 0\\ 0 & -2T_{22}\end{bmatrix} + \left(\lambda + \kappa\left(T_{11} + T_{22}\right)\right)\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix} = \mu\dot{\gamma}(t)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$\Rightarrow \dot{\boldsymbol{T}} = -\left(\lambda + \kappa\left(T_{11} + T_{22}\right)\right)\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix} + \dot{\gamma}(t)\left(\mu\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \begin{bmatrix}T_{11} & 0\\ 0 & -T_{22}\end{bmatrix}\right)$$

$$\Rightarrow \begin{bmatrix}\dot{T}_{11} & \dot{T}_{12}\\ \dot{T}_{12} & \dot{T}_{22}\end{bmatrix} = -\left(\lambda + \kappa\left(T_{11} + T_{22}\right)\right)\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix} + \begin{bmatrix}\mu + T_{11} & 0\\ 0 & -\left(\mu + T_{22}\right)\end{bmatrix}\dot{\gamma}(t) \qquad (3)$$

The system's components are as follows:

$$\Rightarrow \dot{T}_{11} = -\left(\lambda + \kappa\left(T_{11} + T_{22}\right)\right)T_{11} + \left(\mu + T_{11}\right)\dot{\gamma}(t)$$

$$\Rightarrow \dot{T}_{22} = -\left(\lambda + \kappa\left(T_{11} + T_{22}\right)\right)T_{22} - \left(\mu + T_{22}\right)\dot{\gamma}(t)$$

$$\Rightarrow \dot{T}_{12} = -\left(\lambda + \kappa\left(T_{11} + T_{22}\right)\right)T_{12}$$

For mathematical convenience, the following notion is introduced:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix}$$

System (3) can be rewritten as:

$$\frac{d\vec{x}}{dt} = \vec{f}\left(\vec{x}\right) + \vec{g}\left(\vec{x}\right) u$$

where $\vec{f}\left(\vec{x}\right) = \begin{bmatrix} -\left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_1 \\ -\left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_2 \\ -\left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_3 \end{bmatrix}, \quad \vec{g}\left(\vec{x}\right) = \begin{bmatrix} \mu + x_1 \\ -\left(\mu + x_2\right) \\ 0 \end{bmatrix}, \quad u = \dot{\gamma}\left(t\right)$

$$\Rightarrow \begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix} = \begin{bmatrix} -\left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_1 \\ -\left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_2 \\ -\left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_3 \end{bmatrix} + \begin{bmatrix} \mu + x_1 \\ -\left(\mu + x_2\right) \\ 0 \end{bmatrix} \dot{\gamma}\left(t\right)$$

The following Lie Brackets are computed: $\left[\vec{f}, \vec{g}\right]$ and $\left[\vec{f}, \left[\vec{f}, \vec{g}\right]\right]$.

First Lie bracket: $\left[\vec{f}, \vec{g}\right] = \nabla\vec{g} \cdot \vec{f} - \nabla\vec{f} \cdot \vec{g}$

$$\Rightarrow \left[\vec{f}, \vec{g}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -\left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_1 \\ -\left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_2 \\ -\left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_3 \end{bmatrix}$$
$$- \begin{bmatrix} -\left(\lambda + \kappa\left(2x_1 + x_2\right)\right) & -\kappa x_1 & 0 \\ -\kappa x_2 & -\left(\lambda + \kappa\left(x_1 + 2x_2\right)\right) & 0 \\ -\kappa x_3 & -\kappa x_3 & -\left(\lambda + \kappa\left(x_1 + x_2\right)\right) \end{bmatrix} \cdot \begin{bmatrix} \mu + x_1 \\ -\mu - x_2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\vec{f}, \vec{g}\right] = \begin{bmatrix} -\left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_1 \\ \left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_2 \\ 0 \end{bmatrix} - \begin{bmatrix} -\left(\lambda + \kappa\left(2x_1 + x_2\right)\right)\left(\mu + x_1\right) - \kappa x_1\left(-\mu - x_2\right) \\ -\kappa x_2\left(\mu + x_1\right) - \left(\lambda + \kappa\left(x_1 + 2x_2\right)\right)\left(-\mu - x_2\right) \\ -\kappa x_3\left(\mu + x_1\right) - \kappa x_3\left(-\mu - x_2\right) \end{bmatrix}$$

$$\Rightarrow \left[\vec{f}, \vec{g}\right] = \begin{bmatrix} -\left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_1 + \left(\lambda + \kappa\left(2x_1 + x_2\right)\right)\left(\mu + x_1\right) + \kappa x_1\left(-\mu - x_2\right) \\ \left(\lambda + \kappa\left(x_1 + x_2\right)\right) x_2 + \kappa x_2\left(\mu + x_1\right) + \left(\lambda + \kappa\left(x_1 + 2x_2\right)\right)\left(-\mu - x_2\right) \\ \kappa x_3\left(\mu + x_1\right) + \kappa x_3\left(-\mu - x_2\right) \end{bmatrix}$$

$$\Rightarrow\left[\vec{f},\vec{g}\right]=\begin{bmatrix}\kappa x_1\left(x_1-x_2\right)+\mu\left(\lambda+\kappa x_1+\kappa x_2\right)\\ \kappa x_2\left(x_1-x_2\right)-\mu\left(\lambda+\kappa x_1+\kappa x_2\right)\\ \kappa\left(x_1-x_2\right)x_3\end{bmatrix} \tag{4}$$

Second Lie bracket: $\left[\vec{f},\left[\vec{f},\vec{g}\right]\right]=\nabla\left[\vec{f},\vec{g}\right]\cdot\vec{f}-\nabla\vec{f}\cdot\left[\vec{f},\vec{g}\right]$

$$\Rightarrow\left[\vec{f},\left[\vec{f},\vec{g}\right]\right]=\nabla\begin{bmatrix}\kappa x_1^2+\mu\left(\lambda+\kappa x_1+\kappa x_2\right)-\kappa x_1x_2\\ -\kappa x_2^2-\mu\left(\lambda+\kappa x_1+\kappa x_2\right)+\kappa x_1x_2\\ \kappa\left(x_1-x_2\right)x_3\end{bmatrix}\cdot\begin{bmatrix}-\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_1\\ -\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_2\\ -\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_3\end{bmatrix}$$
$$-\nabla\begin{bmatrix}-\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_1\\ -\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_2\\ -\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_3\end{bmatrix}\cdot\begin{bmatrix}\kappa x_1^2+\mu\left(\lambda+\kappa x_1+\kappa x_2\right)-\kappa x_1x_2\\ -\kappa x_2^2-\mu\left(\lambda+\kappa x_1+\kappa x_2\right)+\kappa x_1x_2\\ \kappa\left(x_1-x_2\right)x_3\end{bmatrix}$$

$$=\begin{bmatrix}2\kappa x_1+\mu\kappa-\kappa x_2 & \mu\kappa-\kappa x_1 & 0\\ -\mu\kappa+\kappa x_2 & -2\kappa x_2-\mu\kappa+\kappa x_1 & 0\\ \kappa x_3 & -\kappa x_3 & \kappa\left(x_1-x_2\right)\end{bmatrix}\cdot\begin{bmatrix}-\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_1\\ -\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_2\\ -\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_3\end{bmatrix}$$
$$+\begin{bmatrix}\left(\lambda+\kappa\left(2x_1+x_2\right)\right) & \kappa x_1 & 0\\ \kappa x_2 & \left(\lambda+\kappa\left(x_1+2x_2\right)\right) & 0\\ \kappa x_3 & \kappa x_3 & \left(\lambda+\kappa\left(x_1+x_2\right)\right)\end{bmatrix}\cdot\begin{bmatrix}\kappa x_1\left(x_1-x_2\right)+\mu\left(\lambda+\kappa x_1+\kappa x_2\right)\\ \kappa x_2\left(x_1-x_2\right)-\mu\left(\lambda+\kappa x_1+\kappa x_2\right)\\ \kappa x_3\left(x_1-x_2\right)\end{bmatrix}$$

$$\Rightarrow\left[\vec{f},\left[\vec{f},\vec{g}\right]\right]=-\kappa\left(\lambda+\kappa\left(x_1+x_2\right)\right)\begin{bmatrix}\left(2x_1+\mu-x_2\right)x_1+\left(\mu-x_1\right)x_2\\ \left(-\mu+x_2\right)x_1+\left(-2x_2-\mu+x_1\right)x_2\\ x_1x_3-x_2x_3+\left(x_1-x_2\right)x_3\end{bmatrix}$$
$$+\begin{bmatrix}\kappa x_1\left(x_1-x_2\right)\left[\lambda+2\kappa\left(x_1+x_2\right)\right]+\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right)^2\\ \kappa x_2\left(x_1-x_2\right)\left[\lambda+2\kappa\left(x_2+x_1\right)\right]-\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right)^2\\ \kappa x_3\left(x_1-x_2\right)\left[\lambda+2\kappa\left(x_1+x_2\right)\right]\end{bmatrix}$$

$$\Rightarrow\left[\vec{f},\left[\vec{f},\vec{g}\right]\right]=-\kappa\left(\lambda+\kappa\left(x_1+x_2\right)\right)\begin{bmatrix}2x_1\left(x_1-x_2\right)+\mu\left(x_1+x_2\right)\\ 2x_2\left(x_1-x_2\right)-\mu\left(x_1+x_2\right)\\ 2x_3\left(x_1-x_2\right)\end{bmatrix}$$
$$+\begin{bmatrix}\kappa x_1\left(x_1-x_2\right)\left[\lambda+2\kappa\left(x_1+x_2\right)\right]+\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right)^2\\ \kappa x_2\left(x_1-x_2\right)\left[\lambda+2\kappa\left(x_2+x_1\right)\right]-\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right)^2\\ \kappa x_3\left(x_1-x_2\right)\left[\lambda+2\kappa\left(x_1+x_2\right)\right]\end{bmatrix}$$

$$\Rightarrow \left[\vec{f},\left[\vec{f},\vec{g}\right]\right] = -\kappa\left(\lambda+\kappa\left(x_1+x_2\right)\right)\begin{bmatrix} x_1\left(x_1-x_2\right)+\mu\left(x_1+x_2\right) \\ x_2\left(x_1-x_2\right)-\mu\left(x_1+x_2\right) \\ x_3\left(x_1-x_2\right) \end{bmatrix} + \begin{bmatrix} \kappa^2 x_1\left(x_1+x_2\right)\left(x_1-x_2\right)+\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right)^2 \\ \kappa^2 x_2\left(x_1+x_2\right)\left(x_1-x_2\right)-\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right)^2 \\ \kappa^2 x_3\left(x_1+x_2\right)\left(x_1-x_2\right) \end{bmatrix}$$

$$\Rightarrow \left[\vec{f},\left[\vec{f},\vec{g}\right]\right] = \begin{bmatrix} -\kappa\left(\lambda+\kappa\left(x_1+x_2\right)\right)\left(x_1\left(x_1-x_2\right)-\frac{\mu}{\kappa}\lambda\right)+\kappa^2 x_1\left(x_1+x_2\right)\left(x_1-x_2\right) \\ -\kappa\left(\lambda+\kappa\left(x_1+x_2\right)\right)\left(x_2\left(x_1-x_2\right)+\frac{\mu}{\kappa}\lambda\right)+\kappa^2 x_2\left(x_1+x_2\right)\left(x_1-x_2\right) \\ -\kappa\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_3\left(x_1-x_2\right)+\kappa^2 x_3\left(x_1+x_2\right)\left(x_1-x_2\right) \end{bmatrix}$$

$$\Rightarrow \left[\vec{f},\left[\vec{f},\vec{g}\right]\right] = \begin{bmatrix} -\kappa\lambda x_1\left(x_1-x_2\right)+\mu\lambda\left(\lambda+\kappa\left(x_1+x_2\right)\right) \\ -\kappa\lambda x_2\left(x_1-x_2\right)-\mu\lambda\left(\lambda+\kappa\left(x_1+x_2\right)\right) \\ -\kappa\lambda x_3\left(x_1-x_2\right) \end{bmatrix} \tag{5}$$

Utilizing the vectors from $\vec{g}\left(\vec{x}\right)$, (4) and (5), a matrix is constructed. By investigating the determinant of this matrix, the controllability may be characterized. If the matrix is non-singular, then the system is weakly controllable. Conversely, if the determinant is singular (i.e. equals zero), then the characterization is undetermined and the Lie bracket method offers no insight to the controllability of the system.

Compute the determinant: $\det\left[\vec{g},\left[\vec{f},\vec{g}\right],\left[\vec{f}\left[\vec{f},\vec{g}\right]\right]\right]$.

$$\Rightarrow \det\begin{bmatrix} \mu+x_1 & \kappa x_1\left(x_1-x_2\right)+\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right) & -\kappa\lambda x_1\left(x_1-x_2\right)+\mu\lambda\left(\lambda+\kappa\left(x_1+x_2\right)\right) \\ -\left(\mu+x_2\right) & \kappa x_2\left(x_1-x_2\right)-\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right) & -\kappa\lambda x_2\left(x_1-x_2\right)-\mu\lambda\left(\lambda+\kappa\left(x_1+x_2\right)\right) \\ 0 & \kappa x_3\left(x_1-x_2\right) & -\kappa\lambda x_3\left(x_1-x_2\right) \end{bmatrix}$$

$$
\begin{aligned}
&= (\mu + x_1)(\kappa x_3 (x_1 - x_2)) \left\{ \begin{aligned} &-\lambda\left(\kappa x_2 (x_1 - x_2) - \mu\left(\lambda + \kappa(x_1 + x_2)\right)\right) \\ &\qquad\qquad -\left(-\kappa\lambda x_2 (x_1 - x_2) - \mu\lambda\left(\lambda + \kappa(x_1 + x_2)\right)\right) \end{aligned} \right\} \\
&\quad + (\mu + x_2)(\kappa x_3 (x_1 - x_2)) \left\{ \begin{aligned} &-\lambda\left(\kappa x_1 (x_1 - x_2) + \mu\left(\lambda + \kappa(x_1 + x_2)\right)\right) \\ &\qquad\qquad -\left(-\kappa\lambda x_1 (x_1 - x_2) + \mu\lambda\left(\lambda + \kappa(x_1 + x_2)\right)\right) \end{aligned} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= (\mu + x_1)(\kappa x_3 (x_1 - x_2))\left[2\mu\lambda\left(\lambda + \kappa(x_1 + x_2)\right)\right] \\
&\qquad + (\mu + x_2)(\kappa x_3 (x_1 - x_2))\left[-2\mu\lambda\left(\lambda + \kappa(x_1 + x_2)\right)\right]
\end{aligned}
$$

$$
= \kappa x_3 (x_1 - x_2)(x_1 - x_2)\left[2\mu\lambda\left(\lambda + \kappa(x_1 + x_2)\right)\right]
$$

$$
\Rightarrow \det\left[\vec{g}, \left[\vec{f}, \vec{g}\right], \left[\vec{f}\left[\vec{f}, \vec{g}\right]\right]\right] = 2\mu\lambda\kappa x_3 \left(\lambda + \kappa(x_1 + x_2)\right)(x_1 - x_2)^2
$$

The determinant does not equal zero where $x_3 \neq 0$, $x_1 + x_2 = -\frac{\lambda}{\kappa}$ and $x_1 \neq x_2$. It can be concluded that the system of the Phan-Thien-Tanner model under extensional flow satisfies Definition 3 and thus by Theorem 1, it is weakly controllable when $x_3 \neq 0$ (or $T_{12} \neq 0$) and $x_1 + x_2 = -\frac{\lambda}{\kappa}$ (or $T_{11} + T_{22} \neq -\frac{\lambda}{\kappa}$) and $x_1 \neq x_2$ (or $T_{11} \neq T_{22}$). This result can be summarized geometrically:

Given $R^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in R\}$, there exists three defined surfaces:

$$
\begin{aligned}
S_1 &= \{(x_1, x_2, x_3) \mid x_3 = 0\} \\
S_2 &= \{(x_1, x_2, x_3) \mid \lambda + \kappa(x_1 + x_2) = 0\} \\
S_3 &= \{(x_1, x_2, x_3) \mid x_1 = x_2\}
\end{aligned}
$$

such that the system is weakly controllable at all points in $R^3 \setminus \{S_1 \cup S_2 \cup S_3\}$.

B. HOMOGENOUS SHEAR FLOW

A fluid in a homogeneous shear flow with rate $\dot{\gamma}(t)$ is defined by velocity $\boldsymbol{v} = (\dot{\gamma}(t)y, 0)$, where the velocity gradient tensor is $\nabla \boldsymbol{v} = \dot{\gamma}(t)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Recall, a schematic plot of shear flow is given in Figure 2 of section I. This velocity is applied to the Phan-Thien-Tanner system (2), whereby the system's components are investigated.

$$\Rightarrow \dot{\boldsymbol{T}} - (\nabla \boldsymbol{v})\boldsymbol{T} - \boldsymbol{T}(\nabla \boldsymbol{v})^T + \lambda \boldsymbol{T} + \kappa(tr\boldsymbol{T})\boldsymbol{T} = \mu\left[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\right]$$

$$\Rightarrow \dot{\boldsymbol{T}} - \dot{\gamma}(t)\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{T} + \boldsymbol{T}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) + (\lambda + \kappa(tr\boldsymbol{T}))\boldsymbol{T} = \mu\dot{\gamma}(t)\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)$$

$$\Rightarrow \begin{bmatrix} \dot{T}_{11} & \dot{T}_{12} \\ \dot{T}_{21} & \dot{T}_{22} \end{bmatrix} - \dot{\gamma}(t)\left(\begin{bmatrix} T_{12} & T_{22} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} T_{12} & 0 \\ T_{22} & 0 \end{bmatrix}\right) + (\lambda + \kappa(T_{11} + T_{22}))\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \mu\dot{\gamma}(t)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{T}_{11} & \dot{T}_{12} \\ \dot{T}_{21} & \dot{T}_{22} \end{bmatrix} = -(\lambda + \kappa(T_{11} + T_{22}))\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} + \left(\mu\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2T_{12} & T_{22} \\ T_{22} & 0 \end{bmatrix}\right)\dot{\gamma}(t) \tag{6}$$

The system's components are as follows:

$$\begin{aligned} &\Rightarrow \dot{T}_{11} = -(\lambda + \kappa(T_{11} + T_{22}))T_{11} + 2T_{12}\dot{\gamma}(t) \\ &\Rightarrow \dot{T}_{22} = -(\lambda + \kappa(T_{11} + T_{22}))T_{22} \\ &\Rightarrow \dot{T}_{12} = -(\lambda + \kappa(T_{11} + T_{22}))T_{12} + (\mu + T_{22})\dot{\gamma}(t) \end{aligned} \tag{7}$$

For mathematical convenience, the following notion is introduced:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix}$$

System (6) can be rewritten as:

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) + \vec{g}(\vec{x})u$$

Where $\vec{f}(\vec{x}) = \begin{bmatrix} -(\lambda + \kappa(x_1 + x_2))x_1 \\ -(\lambda + \kappa(x_1 + x_2))x_2 \\ -(\lambda + \kappa(x_1 + x_2))x_3 \end{bmatrix}, \quad \vec{g}(\vec{x}) = \begin{bmatrix} 2x_3 \\ 0 \\ \mu + x_2 \end{bmatrix}, \quad u = \dot{\gamma}(t)$

$$\Rightarrow \begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix} = \begin{bmatrix} -(\lambda + \kappa(x_1 + x_2))x_1 \\ -(\lambda + \kappa(x_1 + x_2))x_2 \\ -(\lambda + \kappa(x_1 + x_2))x_3 \end{bmatrix} + \begin{bmatrix} 2x_3 \\ 0 \\ \mu + x_2 \end{bmatrix} \dot{\gamma}(t)$$

The following Lie Brackets are computed: $\left[\vec{f}, \vec{g}\right]; \left[\vec{f}, \left[\vec{f}, \vec{g}\right]\right]$.

First Lie bracket: $\left[\vec{f}, \vec{g}\right] = \nabla \vec{g} \cdot \vec{f} - \nabla \vec{f} \cdot \vec{g}$

$$\Rightarrow \left[\vec{f}, \vec{g}\right] = \nabla \begin{bmatrix} 2x_3 \\ 0 \\ \mu + x_2 \end{bmatrix} \cdot \begin{bmatrix} -(\lambda + \kappa(x_1 + x_2))x_1 \\ -(\lambda + \kappa(x_1 + x_2))x_2 \\ -(\lambda + \kappa(x_1 + x_2))x_3 \end{bmatrix} - \nabla \begin{bmatrix} -(\lambda + \kappa(x_1 + x_2))x_1 \\ -(\lambda + \kappa(x_1 + x_2))x_2 \\ -(\lambda + \kappa(x_1 + x_2))x_3 \end{bmatrix} \cdot \begin{bmatrix} 2x_3 \\ 0 \\ \mu + x_2 \end{bmatrix}$$

$$\Rightarrow \left[\vec{f}, \vec{g}\right] = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -(\lambda + \kappa(x_1 + x_2))x_1 \\ -(\lambda + \kappa(x_1 + x_2))x_2 \\ -(\lambda + \kappa(x_1 + x_2))x_3 \end{bmatrix}$$
$$+ \begin{bmatrix} (\lambda + \kappa(2x_1 + x_2)) & \kappa x_1 & 0 \\ \kappa x_2 & (\lambda + \kappa(x_1 + 2x_2)) & 0 \\ \kappa x_3 & \kappa x_3 & (\lambda + \kappa(x_1 + x_2)) \end{bmatrix} \cdot \begin{bmatrix} 2x_3 \\ 0 \\ \mu + x_2 \end{bmatrix}$$

$$\Rightarrow \left[\vec{f}, \vec{g}\right] = \begin{bmatrix} -2(\lambda + \kappa(x_1 + x_2))x_3 \\ 0 \\ -(\lambda + \kappa(x_1 + x_2))x_2 \end{bmatrix} + \begin{bmatrix} 2x_3(\lambda + \kappa(2x_1 + x_2)) \\ 2\kappa x_2 x_3 \\ 2\kappa x_3^2 + (\mu + x_2)(\lambda + \kappa(x_1 + x_2)) \end{bmatrix}$$

$$\Rightarrow \left[\vec{f}, \vec{g}\right] = \begin{bmatrix} 0 \\ 2\kappa x_2 x_3 \\ 2\kappa x_3^2 + \mu(\lambda + \kappa(x_1 + x_2)) \end{bmatrix} \tag{8}$$

Second Lie bracket: $\left[\vec{f}, \left[\vec{f}, \vec{g}\right]\right] = \nabla\left[\vec{f}, \vec{g}\right] \cdot \vec{f} - \nabla \vec{f} \cdot \left[\vec{f}, \vec{g}\right]$

$$\Rightarrow\left[\vec{f},\left[\vec{f},\vec{g}\right]\right]=\nabla\begin{bmatrix}0\\2\kappa x_2x_3\\2\kappa x_3^2+\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right)\end{bmatrix}\cdot\begin{bmatrix}-\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_1\\-\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_2\\-\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_3\end{bmatrix}$$

$$-\nabla\begin{bmatrix}-\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_1\\-\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_2\\-\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_3\end{bmatrix}\cdot\begin{bmatrix}0\\2\kappa x_2x_3\\2\kappa x_3^2+\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right)\end{bmatrix}$$

$$\Rightarrow\left[\vec{f},\left[\vec{f},\vec{g}\right]\right]=\begin{bmatrix}0&0&0\\0&2\kappa x_3&2\kappa x_2\\\mu\kappa&\mu\kappa&4\kappa x_3\end{bmatrix}\cdot\begin{bmatrix}-\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_1\\-\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_2\\-\left(\lambda+\kappa\left(x_1+x_2\right)\right)x_3\end{bmatrix}$$

$$+\begin{bmatrix}\left(\lambda+\kappa\left(2x_1+x_2\right)\right)&\kappa x_1&0\\\kappa x_2&\left(\lambda+\kappa\left(x_1+2x_2\right)\right)&0\\\kappa x_3&\kappa x_3&\left(\lambda+\kappa\left(x_1+x_2\right)\right)\end{bmatrix}\cdot\begin{bmatrix}0\\2\kappa x_2x_3\\2\kappa x_3^2+\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right)\end{bmatrix}$$

$$\Rightarrow\left[\vec{f},\left[\vec{f},\vec{g}\right]\right]=-\kappa\left(\lambda+\kappa\left(x_1+x_2\right)\right)\begin{bmatrix}0\\2x_2x_3\\\mu x_1+\mu x_2+4x_3^2\end{bmatrix}$$

$$+\begin{bmatrix}2\kappa^2x_1x_2x_3\\2\kappa x_2x_3\left(\lambda+\kappa\left(x_1+2x_2\right)\right)\\2\kappa^2x_2x_3^2+\left(\lambda+\kappa\left(x_1+x_2\right)\right)\left(2\kappa x_3^2+\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right)\right)\end{bmatrix}$$

$$\Rightarrow\left[\vec{f},\left[\vec{f},\vec{g}\right]\right]=\begin{bmatrix}2\kappa^2x_1x_2x_3\\0\\-2\kappa x_3^2\left(\lambda+\kappa x_1\right)+\mu\lambda\end{bmatrix}\tag{9}$$

Construct a matrix with the vectors from $\vec{g}\left(\vec{x}\right)$, (8) and (9) to compute the determinant: $\det\left[\vec{g},\left[\vec{f},\vec{g}\right],\left[\vec{f}\left[\vec{f},\vec{g}\right]\right]\right]$.

$$\Rightarrow\det\begin{bmatrix}2x_3&0&2\kappa^2x_1x_2x_3\\0&2\kappa x_2x_3&0\\\mu+x_2&2\kappa x_3^2+\mu\left(\lambda+\kappa\left(x_1+x_2\right)\right)&-2\kappa x_3^2\left(\lambda+\kappa x_1\right)+\mu\lambda\end{bmatrix}$$

$$= 4\kappa x_2 x_3^2 \left[-2\lambda\kappa x_3 x_3 + \mu\lambda - \kappa^2 x_1 x_2 \left(\mu + x_2 \right) - 2\kappa^2 x_1 x_3^2 \right]$$

$$\Rightarrow \det\left[\vec{g}, \left[\vec{f}, \vec{g} \right], \left[\vec{f} \left[\vec{f}, \vec{g} \right] \right] \right] = 4\kappa x_2 x_3^2 \left[\mu\lambda - \kappa\left(\kappa x_1 x_2 \left[\mu + x_2 \right] + 2x_3^2 \left[\lambda + \kappa x_1 \right] \right) \right]$$

The determinant does not equal zero where $x_2 \neq 0$, $x_3 \neq 0$, and $\kappa x_1 x_2 \left[\mu + x_2 \right] + 2x_3^2 \left[\lambda + \kappa x_1 \right] = \frac{\mu\lambda}{\kappa}$. It can be concluded that the system of the Phan-Thien-Tanner model under shear flow satisfies Definition 3 and thus by Theorem 1, it is weakly controllable when $x_2 \neq 0$ (or $T_{22} \neq 0$) and $x_3 \neq 0$ (or $T_{12} \neq 0$) and $\kappa x_1 x_2 \left[\mu + x_2 \right] + 2x_3^2 \left[\lambda + \kappa x_1 \right] = \frac{\mu\lambda}{\kappa}$ (or $\kappa T_{11} T_{22} \left[\mu + T_{22} \right] + 2T_{12}^2 \left[\lambda + \kappa T_{11} \right] = \frac{\mu\lambda}{\kappa}$). This result can be summarized geometrically:

Given $R^3 = \left\{ \left(x_1, x_2, x_3 \right) | \, x_1, x_2, x_3 \in R \right\}$, there exists three defined surfaces:

$$S_1 = \left\{ \left(x_1, x_2, x_3 \right) | \, x_2 = 0 \right\}$$
$$S_2 = \left\{ \left(x_1, x_2, x_3 \right) | \, x_3 = 0 \right\}$$
$$S_3 = \left\{ \left(x_1, x_2, x_3 \right) | \, \mu\lambda - \kappa\left(\kappa x_1 x_2 \left[\mu + x_2 \right] + 2x_3^2 \left[\lambda + \kappa x_1 \right] \right) = 0 \right\}$$

such that the system is weakly controllable at all points in $R^3 \setminus \left\{ S_1 \cup S_2 \cup S_3 \right\}$.

This correlates with the work of Renardy [1]. By manipulating the system from (7), it was found that the set of reachable states is given by:

$$S = \left\{ \left(T_{11}\left(t_f \right), T_{12}\left(t_f \right) \right) | \, \mu T_{11}\left(t_f \right) - T_{12}^2\left(t_f \right) > \frac{\lambda\mu}{Ce^{\lambda t_f} - \kappa} \right\}$$

where C is a constant given by:

$$C = \frac{\lambda\mu}{\mu T_{11}\left(0 \right) - T_{12}{}^2\left(0 \right)} + \kappa$$

If $T_{12}\left(0 \right) = 0$, then the reachable set includes the boundary point $\left(T_{11}\left(t_f \right), T_{12}\left(t_f \right) \right) = \left(\frac{\lambda}{Ce^{\lambda t_f} - \kappa}, 0 \right)$.

The result obtained by Renardy is stronger than the one computed with the Lie Brackets in the sense that his result is on the global control whereas the later is on the local control. As mentioned earlier, weak controllability is practically more useful in reducing uncertainties and noises.

C. SUMMARY OF THE PHAN-THIEN-TANNER RESULTS

The Phan-Thien-Tanner model under both homogeneous extensional flow and shear flow, was shown to be weakly controllable on certain submanifolds via the Lie bracket method. Though Renardy [1] had already shown that the Phan-Thien-Tanner model under shear flow could be characterized, this thesis extended the concept by utilizing a different form of analysis, the Lie bracket method. Furthermore, the concept was broadened through the introduction and investigation of an additional flow rate, the homogeneous extensional flow.

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III. JOHNSON-SEGALMAN MODEL

The Johnson-Segalman model characterizes the behavior of non-Newtonian fluids, including special cases of Newtonian and Maxwell fluids. Additionally, it is a viscoelastic fluid model which was developed to allow nonaffline deformations [6].

This model has been used successfully and can be used to explain “spurt” phenomenon (a large increase in volume for a small increase in the driving pressure gradient, at a critical pressure gradient) [7].

The Johnson-Segalman model gives a viscoelastic constitutive equation and can be described by the following state equation:

$$\dot{\boldsymbol{T}} - \frac{a+1}{2}\left(\left(\nabla \boldsymbol{v}\right)\boldsymbol{T} + \boldsymbol{T}\left(\nabla \boldsymbol{v}\right)^T\right) - \frac{a-1}{2}\left(\left(\nabla \boldsymbol{v}\right)^T \boldsymbol{T} + \boldsymbol{T}\left(\nabla \boldsymbol{v}\right)\right) + \lambda \boldsymbol{T} = 2\mu \boldsymbol{D} \qquad (10)$$

where the definition of its element are as follows:

$\boldsymbol{T}$: Stress tensor $\Rightarrow \boldsymbol{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ (where $T_{12} = T_{21}$ due to symmetry)

$\dot{\boldsymbol{T}}$: Derivative of the stress tensor with respect to time

a : Parameter describing polymer slip (a measure of the nonaffinity of the polymer deformation),where $-1 < a < 1$. Note: for the case where $a = 1$, the model reduces to the Oldroyd-B model [8].

$\boldsymbol{v}$: Velocity where $\boldsymbol{v} = \left(v_1, v_2\right)$

λ : Relaxation rate

μ : Elastic modulus

$\boldsymbol{D}$: Rate of deformation tensor (the symmetric part of $\nabla \boldsymbol{v}$) $\Rightarrow \boldsymbol{D} = \dfrac{\nabla \boldsymbol{v} + \left(\nabla \boldsymbol{v}\right)^T}{2}$

$\dot{\gamma}$: Control input (closely related to velocity)

A. HOMOGENOUS EXTENSIONAL FLOW

A fluid in a homogeneous extensional flow with rate $\dot{\gamma}(t)$ is defined by velocity $\boldsymbol{v}=\left(\dot{\gamma}(t)\frac{x}{2},-\dot{\gamma}(t)\frac{y}{2}\right)$, where the velocity gradient tensor is $\nabla\boldsymbol{v}=\frac{\dot{\gamma}(t)}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

This velocity is applied to the system (10), whereby the system's components are investigated:

$$\Rightarrow\begin{bmatrix}\dot{T}_{11} & \dot{T}_{12}\\ \dot{T}_{12} & \dot{T}_{22}\end{bmatrix}-\frac{a+1}{2}\left(\frac{\dot{\gamma}(t)}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix}+\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix}\frac{\dot{\gamma}(t)}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)$$
$$-\frac{a-1}{2}\left(\frac{\dot{\gamma}(t)}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix}+\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix}\frac{\dot{\gamma}(t)}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)+\lambda\boldsymbol{T}$$
$$=\mu\left(\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T\right)$$

$$\Rightarrow\begin{bmatrix}\dot{T}_{11} & \dot{T}_{12}\\ \dot{T}_{12} & \dot{T}_{22}\end{bmatrix}-\frac{\dot{\gamma}(t)}{4}\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix}+\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)\left[(a+1)+(a-1)\right]+\lambda\boldsymbol{T}$$
$$=\mu\left(\frac{\dot{\gamma}(t)}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}+\frac{\dot{\gamma}(t)}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)$$

$$\Rightarrow\begin{bmatrix}\dot{T}_{11} & \dot{T}_{12}\\ \dot{T}_{12} & \dot{T}_{22}\end{bmatrix}-\frac{a\dot{\gamma}(t)}{2}\left(\begin{bmatrix}T_{11} & T_{12}\\ -T_{12} & -T_{22}\end{bmatrix}+\begin{bmatrix}T_{11} & -T_{12}\\ T_{12} & -T_{22}\end{bmatrix}\right)+\lambda\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix}=\mu\dot{\gamma}(t)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}\dot{T}_{11} & \dot{T}_{12}\\ \dot{T}_{12} & \dot{T}_{22}\end{bmatrix}=-\lambda\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix}+\dot{\gamma}(t)\left(\mu\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}+\frac{a}{2}\begin{bmatrix}2T_{11} & 0\\ 0 & -2T_{22}\end{bmatrix}\right)$$

$$\Rightarrow\begin{bmatrix}\dot{T}_{11} & \dot{T}_{12}\\ \dot{T}_{12} & \dot{T}_{22}\end{bmatrix}=-\lambda\begin{bmatrix}T_{11} & T_{12}\\ T_{12} & T_{22}\end{bmatrix}+\begin{bmatrix}\mu+aT_{11} & 0\\ 0 & -(\mu+aT_{22})\end{bmatrix}\dot{\gamma}(t) \tag{11}$$

The system's components are as follows:

$$\Rightarrow\dot{T}_{11}=-\lambda T_{11}+(\mu+aT_{11})\dot{\gamma}(t)$$

$$\Rightarrow\dot{T}_{22}=-\lambda T_{22}-(\mu+aT_{22})\dot{\gamma}(t)$$

$$\Rightarrow\dot{T}_{12}+\lambda T_{12}=0 \tag{12}$$

Note that Equation (12) can be solved exactly:

$$\Rightarrow T_{12} = T_{12}\left(0\right)e^{-\lambda t}$$

Consequently, the Johnson-Segalman model is not controllable. Nonetheless, the state space has a stable invariant subspace $T_{12} = 0$. It is this subspace that all trajectories of the system, under any control input, asymptotically move toward the subspace of $T_{12} = 0$. As a result, the decisive behavior of the control system is characterized by the reduced system on the stable subspace,

$$\begin{aligned} \dot{T}_{11} &= -\lambda T_{11} + \left(\mu + aT_{11}\right)\dot{\gamma}\left(t\right) \\ \dot{T}_{22} &= -\lambda T_{22} - \left(\mu + aT_{22}\right)\dot{\gamma}\left(t\right) \end{aligned} \tag{13}$$

For mathematical convenience, the following notion is introduced:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{22} \end{bmatrix}$$

Then system (13) can be rewritten as:

$$\frac{d\vec{x}}{dt} = \vec{f}\left(\vec{x}\right) + \vec{g}\left(\vec{x}\right)u$$

Where $\vec{f}\left(\vec{x}\right) = \begin{bmatrix} -\lambda x_1 \\ -\lambda x_2 \end{bmatrix}, \quad \vec{g}\left(\vec{x}\right) = \begin{bmatrix} \mu + ax_1 \\ -\left(\mu + ax_2\right) \end{bmatrix}, \quad u = \dot{\gamma}\left(t\right)$

$$\Rightarrow \begin{bmatrix} \dot{T}_{11} \\ \dot{T}_{22} \end{bmatrix} = \begin{bmatrix} -\lambda T_{11} \\ -\lambda T_{22} \end{bmatrix} + \begin{bmatrix} \mu + aT_{11} \\ -\left(\mu + aT_{22}\right) \end{bmatrix}\dot{\gamma}\left(t\right)$$

The following Lie Bracket is computed: $\left[\vec{f}, \vec{g}\right]$.

$$\Rightarrow \left[\vec{f}, \vec{g}\right] = \nabla\vec{g} \cdot \vec{f} - \nabla\vec{f} \cdot \vec{g}$$

$$\Rightarrow \left[\vec{f}, \vec{g}\right] = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \cdot \begin{bmatrix} -\lambda x_1 \\ -\lambda x_2 \end{bmatrix} - \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix} \cdot \begin{bmatrix} \mu + ax_1 \\ -\left(\mu + ax_2\right) \end{bmatrix}$$

$$\Rightarrow \left[\vec{f},\vec{g}\right] = \begin{bmatrix} -a\lambda x_1 \\ a\lambda x_2 \end{bmatrix} - \begin{bmatrix} -\lambda\left(\mu + ax_1\right) \\ \lambda\left(\mu + ax_2\right) \end{bmatrix} = \begin{bmatrix} -a\lambda x_1 + \lambda\left(\mu + ax_1\right) \\ a\lambda x_2 - \lambda\left(\mu + ax_2\right) \end{bmatrix} = \begin{bmatrix} \lambda\mu \\ -\lambda\mu \end{bmatrix} \tag{14}$$

Construct a matrix with the vectors from $\vec{g}\left(\vec{x}\right)$ and (14) to compute the determinant: $\det\left[\vec{g},\left[\vec{f},\vec{g}\right]\right]$.

$$\Rightarrow \det\left[\vec{g},\left[\vec{f},\vec{g}\right]\right] = \det\begin{bmatrix} \mu + ax_1 & \lambda\mu \\ -\left(\mu + ax_2\right) & -\lambda\mu \end{bmatrix} = a\lambda\mu\left(x_2 - x_1\right)$$

The determinant does not equal zero where $x_1 \neq x_2$. It can be concluded that since the sub system (13) of the Johnson-Segalman model under extensional flow satisfies Definition 3 and thus by Theorem 1, it is weakly controllable when $x_1 \neq x_2$ (or $T_{11} \neq T_{22}$). This result can be summarized geometrically:

Given $R^2 = \left\{\left(x_1, x_2\right) \mid x_1, x_2 \in R\right\}$, there exists one defined surface:

$S_1 = \left\{\left(x_1, x_2\right) \mid x_1 = x_2\right\}$

such that the system is weakly controllable at all points in $R^2 \setminus \left\{S_1\right\}$.

B. HOMOGENOUS SHEAR FLOW

A fluid in a homogeneous shear flow with rate $\dot{\gamma}\left(t\right)$ is defined by velocity $\boldsymbol{v} = \left(\dot{\gamma}\left(t\right)y, 0\right)$, where the velocity gradient tensor is $\nabla\boldsymbol{v} = \dot{\gamma}\left(t\right)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This velocity is applied to the system (10), whereby the system's components are investigated:

$$\begin{aligned} \Rightarrow \begin{bmatrix} \dot{T}_{11} & \dot{T}_{12} \\ \dot{T}_{12} & \dot{T}_{22} \end{bmatrix} &- \frac{a+1}{2}\left(\dot{\gamma}\left(t\right)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix} + \begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix}\dot{\gamma}\left(t\right)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \\ &- \frac{a-1}{2}\left(\dot{\gamma}\left(t\right)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix} + \begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix}\dot{\gamma}\left(t\right)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) + \lambda\begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix} \\ &= \mu\left(\nabla\boldsymbol{v} + \left(\nabla\boldsymbol{v}\right)^T\right) \end{aligned}$$

$$\Rightarrow \begin{bmatrix} \dot{T}_{11} & \dot{T}_{12} \\ \dot{T}_{12} & \dot{T}_{22} \end{bmatrix} - \frac{(a+1)\dot{\gamma}(t)}{2}\left(\begin{bmatrix} T_{12} & T_{22} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} T_{12} & 0 \\ T_{22} & 0 \end{bmatrix} \right)$$

$$- \frac{(a-1)\dot{\gamma}(t)}{2}\left(\begin{bmatrix} 0 & 0 \\ T_{11} & T_{12} \end{bmatrix} + \begin{bmatrix} 0 & T_{11} \\ 0 & T_{12} \end{bmatrix} \right) + \lambda \begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix}$$

$$= \mu \dot{\gamma}(t) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{T}_{11} & \dot{T}_{12} \\ \dot{T}_{12} & \dot{T}_{22} \end{bmatrix} = -\lambda \begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix}$$

$$+\dot{\gamma}(t)\left(\mu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2}\left((a+1)\begin{bmatrix} 2T_{12} & T_{22} \\ T_{22} & 0 \end{bmatrix} + (a-1)\begin{bmatrix} 0 & T_{11} \\ T_{11} & 2T_{12} \end{bmatrix} \right) \right)$$

$$\Rightarrow \begin{bmatrix} \dot{T}_{11} & \dot{T}_{12} \\ \dot{T}_{12} & \dot{T}_{22} \end{bmatrix} = -\lambda \begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix}$$

$$+ \begin{bmatrix} (a+1)T_{12} & \mu + \frac{1}{2}\left((a-1)T_{11} + (a+1)T_{22}\right) \\ \mu + \frac{1}{2}\left((a-1)T_{11} + (a+1)T_{22}\right) & (a-1)T_{12} \end{bmatrix} \dot{\gamma}(t) \tag{15}$$

The system’s components are as follows:

$$\begin{aligned} &\Rightarrow \dot{T}_{11} = -\lambda T_{11} + (a+1)T_{12}\dot{\gamma}(t) \\ &\Rightarrow \dot{T}_{22} = -\lambda T_{22} + (a-1)T_{12}\dot{\gamma}(t) \\ &\Rightarrow \dot{T}_{12} = -\lambda T_{12} + \left[\mu + \frac{1}{2}\left((a-1)T_{11} + (a+1)T_{22}\right) \right]\dot{\gamma}(t) \end{aligned} \tag{16}$$

For mathematical convenience, the following notion is introduced:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix}$$

System (15) can be rewritten as:

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) + \vec{g}(\vec{x})u$$

Where $\vec{f}(\vec{x}) = \begin{bmatrix} -\lambda x_1 \\ -\lambda x_2 \\ -\lambda x_3 \end{bmatrix}$, $\vec{g}(\vec{x}) = \begin{bmatrix} (a+1)x_3 \\ (a-1)x_3 \\ \mu + \frac{1}{2}\left((a-1)x_1 + (a+1)x_2\right) \end{bmatrix}$, $u = \dot{\gamma}(t)$

$$\Rightarrow \begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix} = \begin{bmatrix} -\lambda x_1 \\ -\lambda x_2 \\ -\lambda x_3 \end{bmatrix} + \begin{bmatrix} (a+1)x_3 \\ (a-1)x_3 \\ \mu + \frac{1}{2}\left((a-1)x_1 + (a+1)x_2\right) \end{bmatrix} \dot{\gamma}(t)$$

We will now compute the following Lie Brackets: $\left[\vec{f},\vec{g}\right]$; $\left[\vec{f},\left[\vec{f},\vec{g}\right]\right]$.

Compute first Lie bracket: $\left[\vec{f},\vec{g}\right] = \nabla\vec{g}\cdot\vec{f} - \nabla\vec{f}\cdot\vec{g}$.

$$\Rightarrow \left[\vec{f},\vec{g}\right] = \nabla \begin{bmatrix} (a+1)x_3 \\ (a-1)x_3 \\ \mu + \frac{1}{2}\left((a-1)x_1 + (a+1)x_2\right) \end{bmatrix} \cdot \begin{bmatrix} -\lambda x_1 \\ -\lambda x_2 \\ -\lambda x_3 \end{bmatrix}$$

$$- \nabla \begin{bmatrix} -\lambda x_1 \\ -\lambda x_2 \\ -\lambda x_3 \end{bmatrix} \cdot \begin{bmatrix} (a+1)x_3 \\ (a-1)x_3 \\ \mu + \frac{1}{2}\left((a-1)x_1 + (a+1)x_2\right) \end{bmatrix}$$

$$\Rightarrow \left[\vec{f},\vec{g}\right] = \begin{bmatrix} 0 & 0 & (a+1) \\ 0 & 0 & (a-1) \\ \frac{1}{2}(a-1) & \frac{1}{2}(a+1) & 0 \end{bmatrix} \cdot \begin{bmatrix} -\lambda x_1 \\ -\lambda x_2 \\ -\lambda x_3 \end{bmatrix}$$

$$- \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} \cdot \begin{bmatrix} (a+1)x_3 \\ (a-1)x_3 \\ \mu + \frac{1}{2}\left((a-1)x_1 + (a+1)x_2\right) \end{bmatrix}$$

$$\Rightarrow \left[\vec{f},\vec{g}\right] = \lambda \begin{bmatrix} -x_3\left(a+1\right) \\ -x_3\left(a-1\right) \\ -\left(\frac{x_1}{2}\left(a-1\right)+\frac{x_2}{2}\left(a+1\right)\right) \end{bmatrix} + \lambda \begin{bmatrix} x_3\left(a+1\right) \\ x_3\left(a-1\right) \\ \mu+\frac{1}{2}\left(\left(a-1\right)x_1+\left(a+1\right)x_2\right) \end{bmatrix}$$

$$\Rightarrow \left[\vec{f},\vec{g}\right] = \begin{bmatrix} 0 \\ 0 \\ \mu\lambda \end{bmatrix} \tag{17}$$

Compute second Lie bracket: $\left[\vec{f},\left[\vec{f},\vec{g}\right]\right] = \nabla\left[\vec{f},\vec{g}\right]\cdot\vec{f} - \nabla\vec{f}\cdot\left[\vec{f},\vec{g}\right]$.

$$\Rightarrow \left[\vec{f},\left[\vec{f},\vec{g}\right]\right] = \nabla\begin{bmatrix} 0 \\ 0 \\ \mu\lambda \end{bmatrix}\cdot\begin{bmatrix} -\lambda x_1 \\ -\lambda x_2 \\ -\lambda x_3 \end{bmatrix} - \nabla\begin{bmatrix} -\lambda x_1 \\ -\lambda x_2 \\ -\lambda x_3 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ \mu\lambda \end{bmatrix} \ \Rightarrow \left[\vec{f},\left[\vec{f},\vec{g}\right]\right] = \begin{bmatrix} 0 \\ 0 \\ \mu\lambda^2 \end{bmatrix} \tag{18}$$

Construct a matrix with the vectors from $\vec{g}\left(\vec{x}\right)$, (17) and (18) to compute the determinant: $\det\left[\vec{g},\left[\vec{f},\vec{g}\right],\left[\vec{f}\left[\vec{f},\vec{g}\right]\right]\right]$.

$$\Rightarrow \det\begin{bmatrix} \left(a+1\right)x_3 & 0 & 0 \\ \left(a-1\right)x_3 & 0 & 0 \\ \mu+\frac{1}{2}\left(\left(a-1\right)x_1+\left(a+1\right)x_2\right) & \mu\lambda & \mu\lambda^2 \end{bmatrix} = 0$$

The determinant equals zero everywhere and thus is singular. It can be concluded that the controllability of the Johnson-Segalman model under shear flow characterization cannot be determined by $\vec{g},\left[\vec{f},\vec{g}\right],$ and $\left[\vec{f}\left[\vec{f},\vec{g}\right]\right]$. Nevertheless, it is possible to obtain control invariant manifolds in the state space that are weakly controllable. This needs more advanced geometric analysis and it is beyond the scope of this thesis.

It should be pointed out that by some clever direct analysis, Renardy [1] was able to derive some complicated expressions for the reachable set. His result can be

summarized as follows. We first introduce a new variable which is a linear combination of the two normal differences T_{11} and T_{22}:

$$Z = \frac{a}{2}\left(T_{11} + T_{22}\right) + \frac{1}{2}\left(T_{22} - T_{11}\right).$$

Then the Johnson-Segalman system can be reduced into a system of ODEs for $\left(T_{12}, Z\right)$:

$$\begin{aligned} \dot{T}_{12} &= -\lambda T_{12} + \left(Z + \mu\right)\dot{\gamma}\left(t\right) \\ \dot{Z} &= -\lambda Z + \left(a^2 - 1\right)T_{12}\dot{\gamma}\left(t\right) \end{aligned}$$

Using arguments analogous to that for the Phan-Thien-Tanner model, Renardy derived the reachable set S defined in the $\left(T_{12}, Z\right)$ plane:

$$S = \left\{ \left(T_{12}\left(t_f\right), Z\left(t_f\right)\right) \middle| \phi_2\left(t_f\right) \le \frac{1}{2}Z\left(t_f\right)^2 + \mu Z\left(t_f\right) + \frac{1}{2}\left(1 - a^2\right)T_{12}\left(t_f\right)^2 \le \phi_1\left(t_f\right) \right\}$$

Here in the inequalities the lower and upper bounds are solutions of the IVPs:

$$\begin{aligned} &\frac{d}{dt}\phi_1\left(t\right) = -2\lambda\phi_1\left(t\right) + \lambda\mu\left[-\mu + \sqrt{2\phi_1\left(t\right) + \mu^2}\right], \\ &\frac{d}{dt}\phi_2\left(t\right) = -2\lambda\phi_2\left(t\right) + \lambda\mu\left[-\mu - \sqrt{2\phi_2\left(t\right) + \mu^2}\right], \\ &\phi_1\left(0\right) = \phi_2\left(0\right) = \frac{1}{2}Z\left(0\right)^2 + \mu Z\left(0\right) + \frac{1}{2}\left(1 - a^2\right)T_{12}\left(0\right)^2 \end{aligned}$$

If $T_{12}\left(0\right) = 0$, then the reachable set includes the boundary point $\left(T_{12}\left(t_f\right), Z\left(t_f\right)\right) = \left(0, -\mu + \sqrt{2\phi_1\left(t_f\right) + \mu^2}\right)$ or $\left(T_{12}\left(t_f\right), Z\left(t_f\right)\right) = \left(0, -\mu - \sqrt{2\phi_2\left(t_f\right) + \mu^2}\right)$.

Again, Renardy's result is mathematically stronger than the one computed from the Lie bracket method whereas the later is more practical.

C. SUMMARY OF THE JOHNSON-SEGALMAN RESULTS

The Johnson-Segalman model with homogeneous extensional flow showed to be weakly controllable for a subsystem, whereas, the shear flow was undetermined. Though Renardy [1] had already shown that the Johnson-Segalman model under shear flow could

be characterized, this thesis extended the concept by utilizing a different form of analysis, the Lie bracket method. Furthermore, the concept was broadened through the introduction and investigation of an additional flow rate, the homogeneous extensional flow.

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IV. DOI MODEL

The Doi model for rod-like molecules is known for its ability to represent the characteristics of liquid crystal polymers in a solvent. The single molecule position-orientation distribution function is a fundamental element of the model. Exchanges between molecules are represented by a mean-field potential and the model can represent mean-field kinetic theory. Rods are subject to Brownian force due to the fact that they interact with other rods and with the flow. Generally, the model is a microscopic Fokker–Planck type equation (Smoluchowski equation depicting the convection, rotation and diffusion of the rods) coupled with a macroscopic Stokes equation (pertaining to the hydrodynamics) [9, 10].

A fundamental attribute of the Doi model is that it has the capability to describe both the isotropic and nematic phases. If the interaction strength of the rods is effectively strong or if the rod concentration is high, then the system tends towards a nematic phase and the rods have a propensity to line up. If the rods randomly orientated, then the system is an isotropic phase [9, 10].

The full Doi orientation tensor theory is developed after the microscopic Fokker-Planck equation is projected onto a second-moment description using closure rules. The major ingredient in this tensor theory is the second-moment tensor which describes the orientational distribution of the ensemble of rod-like macromolecules. The orientation tensor is traceless and symmetric. It is the basis for micron-scale light scattering measurements of primary axes (directors), degrees of molecular alignment (birefringence), and normal and shear stress measurements.

This section is devoted to the controllability of two-dimensional Doi tensor model. The study of two-dimensional liquid is physically motivated by monolayer films. As a result of their stability and nonlinear optical characteristics, thin films of liquid crystalline polymers have drawn a significant amount of attention contrast to materials of low molecular weight. The controllability of this model under homogeneous flows will be explored for the first time in the literature. Since the bulk properties of liquid

crystalline polymers (LCPs) depend on the microstructure, especially the orientation of the molecules, the ability to control orientation tensor will open the door for achieving the desired strength and materials properties of LCPs.

The two-dimensional Doi model for liquid crystalline polymer is described by the following state equation:

$$\dot{\boldsymbol{Q}} = \Omega\boldsymbol{Q} - \boldsymbol{Q}\Omega + a\left[\boldsymbol{DQ} + \boldsymbol{QD}\right] + a\boldsymbol{D} - 2a\boldsymbol{D} : \boldsymbol{Q}\left(\boldsymbol{Q} + \frac{I}{2}\right) - 6D_r^0 F\left(\boldsymbol{Q}\right) \qquad (19)$$

where the definition of its element are as follows:

$\boldsymbol{Q}$: Orientation tensor , $\Rightarrow \boldsymbol{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix}$ (where $-Q_{11} = Q_{22}$ and $Q_{12} = Q_{21}$)

$\dot{\boldsymbol{Q}}$: Derivative of the orientation tensor with respect to time

$\boldsymbol{\Omega}$: Vorticity tensor $\Rightarrow \boldsymbol{\Omega} = \frac{1}{2}\left[\nabla\boldsymbol{v} - \nabla\boldsymbol{v}^T\right]$

$\boldsymbol{v}$: Velocity where $\boldsymbol{v} = \left(v_1, v_2\right)$

a : Dimensionless parameter depending on the molecular aspect ratio,

$\boldsymbol{D}$: Rate-of-strain tensor $\Rightarrow \boldsymbol{D} = \frac{1}{2}\left[\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T\right]$

$$F\left(\boldsymbol{Q}\right) = \left(1 - \frac{N}{2}\right)\boldsymbol{Q} - N\boldsymbol{Q}^2 + N\boldsymbol{Q} : \boldsymbol{Q}\left(\boldsymbol{Q} + \frac{I}{2}\right)$$

$$\Rightarrow F\left(\boldsymbol{Q}\right) = \left(1 - \frac{N}{2}\right)\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} - N\begin{bmatrix} \left(Q_{11}\right)^2 + \left(Q_{12}\right)^2 & 0 \\ 0 & \left(Q_{11}\right)^2 + \left(Q_{12}\right)^2 \end{bmatrix}$$
$$+ 2N\left(\left(Q_{11}\right)^2 + \left(Q_{12}\right)^2\right)\begin{bmatrix} Q_{11} + \frac{1}{2} & Q_{12} \\ Q_{12} & -Q_{11} + \frac{1}{2} \end{bmatrix}$$

N : Dimensionless concentration of nematic polymers,

D_r^0 : Averaged rotary diffusivity or relaxation rate

Notation $\left[\boldsymbol{Q}:\boldsymbol{Q}\right] = tr\left(\boldsymbol{Q}\boldsymbol{Q}^T\right)$

A. HOMOGENOUS EXTENSIONAL FLOW

A fluid in a homogeneous extensional flow with rate $\dot{\gamma}(t)$ is defined by velocity $\boldsymbol{v} = \left(\dot{\gamma}(t)\frac{x}{2}, -\dot{\gamma}(t)\frac{y}{2}\right)$, where the velocity gradient tensor is $\nabla\boldsymbol{v} = \frac{\dot{\gamma}(t)}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

This velocity is applied to the system (19), whereby the system's components are investigated:

$$\Rightarrow \dot{\boldsymbol{Q}} = a\frac{\dot{\gamma}(t)}{2}\left[\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{12} & -Q_{11}\end{bmatrix} + \begin{bmatrix}Q_{11} & Q_{12}\\ Q_{12} & -Q_{11}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right] + a\frac{\dot{\gamma}(t)}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$
$$- a\dot{\gamma}(t)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} : \begin{bmatrix}Q_{11} & Q_{12}\\ Q_{12} & -Q_{11}\end{bmatrix}\left(\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{12} & -Q_{11}\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) - 6D_r^0 F(\boldsymbol{Q})$$

$$\Rightarrow \dot{\boldsymbol{Q}} = a\frac{\dot{\gamma}(t)}{2}\left[\begin{bmatrix}Q_{11} & Q_{12}\\ -Q_{12} & Q_{11}\end{bmatrix} + \begin{bmatrix}Q_{11} & -Q_{12}\\ Q_{12} & Q_{11}\end{bmatrix}\right] + a\frac{\dot{\gamma}(t)}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$
$$- a\dot{\gamma}(t)tr\begin{bmatrix}Q_{11} & Q_{12}\\ -Q_{12} & Q_{11}\end{bmatrix}\left(\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{12} & -Q_{11}\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) - 6D_r^0 F(\boldsymbol{Q})$$

$$\Rightarrow \dot{\boldsymbol{Q}} = a\dot{\gamma}(t)\begin{bmatrix}Q_{11} & 0\\ 0 & Q_{11}\end{bmatrix} + a\frac{\dot{\gamma}(t)}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} - a\dot{\gamma}(t)(2Q_{11})\left(\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{12} & -Q_{11}\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)$$
$$- 6D_r^0\left\{\begin{aligned}&\left(1-\frac{N}{2}\right)\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{12} & -Q_{11}\end{bmatrix} - N\begin{bmatrix}Q_{11}^2+Q_{12}^2 & 0\\ 0 & Q_{11}^2+Q_{12}^2\end{bmatrix}\\ &\qquad + 2N\left(Q_{11}^2+Q_{12}^2\right)\begin{bmatrix}Q_{11}+\frac{1}{2} & Q_{12}\\ Q_{12} & -Q_{11}+\frac{1}{2}\end{bmatrix}\end{aligned}\right\}$$

$$\Rightarrow \dot{\boldsymbol{Q}} = -6D_r^0 \left\{ \begin{array}{l} \left(1-\frac{N}{2}\right)\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} - N \begin{bmatrix} Q_{11}^2+Q_{12}^2 & 0 \\ 0 & Q_{11}^2+Q_{12}^2 \end{bmatrix} \\ \qquad\qquad + 2N\left(Q_{11}^2+Q_{12}^2\right)\begin{bmatrix} Q_{11}+\frac{1}{2} & Q_{12} \\ Q_{12} & -Q_{11}+\frac{1}{2} \end{bmatrix} \end{array} \right\}$$

$$+\left[a\begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{11} \end{bmatrix} + \frac{a}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - 2aQ_{11}\left(\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]\dot{\gamma}(t)$$

$$\Rightarrow \dot{\boldsymbol{Q}} = -6D_r^0 \left\{ \begin{array}{l} \left(1-\frac{N}{2}\right)\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} - N \begin{bmatrix} Q_{11}^2+Q_{12}^2 & 0 \\ 0 & Q_{11}^2+Q_{12}^2 \end{bmatrix} \\ \qquad\qquad + 2N\left(Q_{11}^2+Q_{12}^2\right)\begin{bmatrix} Q_{11}+\frac{1}{2} & Q_{12} \\ Q_{12} & -Q_{11}+\frac{1}{2} \end{bmatrix} \end{array} \right\}$$

$$+a\left[\begin{bmatrix} Q_{11}+\frac{1}{2} & 0 \\ 0 & Q_{11}-\frac{1}{2} \end{bmatrix} - 2Q_{11}\begin{bmatrix} Q_{11}+\frac{1}{2} & Q_{12} \\ Q_{12} & -Q_{11}+\frac{1}{2} \end{bmatrix}\right]\dot{\gamma}(t) \qquad (20)$$

We get the following two component equations:

$$\Rightarrow \dot{Q}_{11} = -6D_r^0\left(\left(1-\frac{N}{2}\right)Q_{11} + 2NQ_{11}\left(Q_{11}^2+Q_{12}^2\right)\right) + a\left(\frac{1}{2} - 2Q_{11}^2\right)\dot{\gamma}(t)$$

$$\Rightarrow \dot{Q}_{12} = -6D_r^0 Q_{12}\left(\left(1-\frac{N}{2}\right) + 2N\left(Q_{11}^2+Q_{12}^2\right)\right) - 2aQ_{11}Q_{12}\dot{\gamma}(t)$$

Regard $\dot{\gamma}(t)$ as a control input and consider the problem (19) with initial state Q_0 and final state Q_1. The system is controllable if for any choices of Q_0 and Q_1 there exists a control $\dot{\gamma}(t)$ on the interval $(0,T)$.

Using the nematic relaxation time scale $\frac{1}{D_r^0}$, the flow field and orientation dynamics of (19) can be non-dimensionalized. The key dimensionless parameters are

then the Peclet number $Pe(t) = \frac{\dot{\gamma}}{D_r^0}$ (the shear rate normalized with respect to nematic relaxation rate) and the dimensionless concentration parameter N.

$$\Rightarrow \dot{Q}_{11} = D_r^0 \left[-6Q_{11}\left(1 - \frac{N}{2} + 2N\left(Q_{11}^2 + Q_{12}^2\right)\right) + a\left(\frac{1}{2} - 2Q_{11}^2\right) Pe(t) \right]$$

$$\Rightarrow \dot{Q}_{12} = D_r^0 \left[-6Q_{12}\left(1 - \frac{N}{2} + 2N\left(Q_{11}^2 + Q_{12}^2\right)\right) - 2aQ_{11}Q_{12} Pe(t) \right]$$

Applying $\bar{t} = tD_r^0$, the equations become dimensionless and the nematodynamic model (19) can be expressed as:

$$\begin{aligned} &\Rightarrow \dot{Q}_{11} = -6Q_{11}\left(1 - \frac{N}{2} + 2N\left(Q_{11}^2 + Q_{12}^2\right)\right) + a\left(\frac{1}{2} - 2Q_{11}^2\right) Pe(t) \\ &\Rightarrow \dot{Q}_{12} = -6Q_{12}\left(1 - \frac{N}{2} + 2N\left(Q_{11}^2 + Q_{12}^2\right)\right) - 2aQ_{11}Q_{12} Pe(t) \end{aligned} \tag{21}$$

For mathematical convenience, the following notion is introduced:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} Q_{11} \\ Q_{12} \end{bmatrix}$$

Then system (21) can be written as:

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) + \vec{g}(\vec{x}) u$$

Where $\vec{f}(\vec{x}) = \begin{bmatrix} -6x_1\left(1 - \frac{N}{2} + 2N\left(x_1^2 + x_2^2\right)\right) \\ -6x_2\left(1 - \frac{N}{2} + 2N\left(x_1^2 + x_2^2\right)\right) \end{bmatrix}$, $\vec{g}(\vec{x}) = \begin{bmatrix} a\left(\frac{1}{2} - 2x_1^2\right) \\ -2ax_1x_2 \end{bmatrix}$, $u = Pe(t)$

Compute the following Lie Bracket: $\left[\vec{f}, \vec{g}\right] = \nabla\vec{g} \cdot \vec{f} - \nabla\vec{f} \cdot \vec{g}$.

$$\left[\vec{f},\vec{g}\right]=\begin{bmatrix}-4ax_1 & 0\\ -2ax_2 & -2ax_1\end{bmatrix}\cdot\begin{bmatrix}-6x_1\left(1-\dfrac{N}{2}+2N\left(x_1^2+x_2^2\right)\right)\\ -6x_2\left(1-\dfrac{N}{2}+2N\left(x_1^2+x_2^2\right)\right)\end{bmatrix}$$

$$-\begin{bmatrix}-6\left(1-\dfrac{N}{2}+2N\left(3x_1^2+x_2^2\right)\right) & -24Nx_1x_2\\ -24Nx_1x_2 & -6\left(1-\dfrac{N}{2}+2N\left(x_1^2+3x_2^2\right)\right)\end{bmatrix}\cdot\begin{bmatrix}a\left(\dfrac{1}{2}-2x_1^2\right)\\ -2ax_1x_2\end{bmatrix}$$

$$\left[\vec{f},\vec{g}\right]=\begin{bmatrix}24ax_1^2\left(1-\dfrac{N}{2}+2N\left(x_1^2+x_2^2\right)\right)\\ 24ax_1x_2\left(1-\dfrac{N}{2}+2N\left(x_1^2+x_2^2\right)\right)\end{bmatrix}$$

$$-\begin{bmatrix}-6a\left(1-\dfrac{N}{2}+2N\left(3x_1^2+x_2^2\right)\right)\left(\dfrac{1}{2}-2x_1^2\right)+48aNx_1^2x_2^2\\ -24aNx_1x_2\left(\dfrac{1}{2}-2x_1^2\right)+12ax_1x_2\left(1-\dfrac{N}{2}+2N\left(x_1^2+3x_2^2\right)\right)\end{bmatrix}$$

$$\left[\vec{f},\vec{g}\right]=\begin{bmatrix}6a\left\{4x_1^2\left(1-\dfrac{N}{2}+2N\left(x_1^2+x_2^2\right)\right)+\left(1-\dfrac{N}{2}+2N\left(3x_1^2+x_2^2\right)\right)\left(\dfrac{1}{2}-2x_1^2\right)-8Nx_1^2x_2^2\right\}\\ 12ax_1x_2\left\{2\left(1-\dfrac{N}{2}+2N\left(x_1^2+x_2^2\right)\right)+2N\left(\dfrac{1}{2}-2x_1^2\right)-\left(1-\dfrac{N}{2}+2N\left(x_1^2+3x_2^2\right)\right)\right\}\end{bmatrix}$$

$$\left[\vec{f},\vec{g}\right]=\begin{bmatrix}6a\left\{4x_1^2\left(1-\dfrac{N}{2}+2Nx_1^2\right)+\left(1-\dfrac{N}{2}+2N\left(3x_1^2+x_2^2\right)\right)\left(\dfrac{1}{2}-2x_1^2\right)\right\}\\ 12ax_1x_2\left\{1+\dfrac{N}{2}-2N\left(x_1^2+x_2^2\right)\right\}\end{bmatrix}$$

$$\left[\vec{f},\vec{g}\right]=\begin{bmatrix}6a\left\{2x_1^2\left(1-\dfrac{N}{2}-2N\left(x_1^2+x_2^2\right)\right)+\dfrac{1}{2}-\dfrac{N}{4}+N\left(3x_1^2+x_2^2\right)\right\}\\ 12ax_1x_2\left\{1+\dfrac{N}{2}-2N\left(x_1^2+x_2^2\right)\right\}\end{bmatrix}\qquad(22)$$

Construct a matrix with the vectors from $\vec{g}\left(\vec{x}\right)$ and (22) to compute the determinant: $\det\left[\vec{g},\left[\vec{f},\vec{g}\right]\right]$.

$$\Rightarrow \det\begin{bmatrix} a\left(\frac{1}{2}-2x_1^2\right) & 6a\left\{2x_1^2\left(1-\frac{N}{2}-2N\left(x_1^2+x_2^2\right)\right)+\frac{1}{2}-\frac{N}{4}+N\left(3x_1^2+x_2^2\right)\right\} \\ -2ax_1x_2 & 12ax_1x_2\left\{1+\frac{N}{2}-2N\left(x_1^2+x_2^2\right)\right\} \end{bmatrix}$$

$$=12a^2x_1x_2\left(\frac{1}{2}-2x_1^2\right)\left\{1+\frac{N}{2}-2N\left(x_1^2+x_2^2\right)\right\}$$
$$+12a^2x_1x_2\left\{2x_1^2\left(1-\frac{N}{2}-2N\left(x_1^2+x_2^2\right)\right)+\frac{1}{2}-\frac{N}{4}+N\left(3x_1^2+x_2^2\right)\right\}$$

$$=12a^2x_1x_2\left[\frac{1}{2}\left\{1+\frac{N}{2}-2N\left(x_1^2+x_2^2\right)\right\}-2Nx_1^2+\frac{1}{2}-\frac{N}{4}+N\left(3x_1^2+x_2^2\right)\right]$$

$$\Rightarrow \det\left[\vec{g},\left[\vec{f},\vec{g}\right]\right]=12a^2x_1x_2$$

The determinant equals zero where $x_1 \neq 0$ and $x_2 \neq 0$. It can be concluded that the controllability of the sub system from the Doi model under extensional flow satisfies Definition 3 and thus by Theorem 1, it is weakly controllable when $x_1 \neq 0$ (or $Q_{11} \neq 0$) and $x_2 \neq 0$ (or $Q_{12} \neq 0$). This result can be summarized geometrically:

Given $R^2 = \left\{\left(x_1, x_2\right) \mid x_1, x_2 \in R\right\}$, there exists two defined surfaces:

$$S_1 = \left\{\left(x_1, x_2\right) \mid x_1 = 0\right\}$$
$$S_2 = \left\{\left(x_1, x_2\right) \mid x_2 = 0\right\}$$

such that the system is weakly controllable at all points in $R^2 \setminus \left\{S_1 \cup S_2\right\}$.

B. HOMOGENOUS SHEAR FLOW

A fluid in a homogeneous shear flow with rate $\dot{\gamma}(t)$ is defined by velocity $\boldsymbol{v} = \left(\dot{\gamma}(t)\, y, 0\right)$, where the velocity gradient tensor is $\nabla \boldsymbol{v} = \dot{\gamma}(t)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

This velocity is applied to the system (19), whereby the system's components are investigated:

$$\Rightarrow \dot{\boldsymbol{Q}} = \frac{\dot{\gamma}(t)}{2}\left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} - \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right]$$

$$+a\frac{\dot{\gamma}(t)}{2}\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] + a\frac{\dot{\gamma}(t)}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$-a\dot{\gamma}(t)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix}\left(\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) - 6D_r^0 F(\boldsymbol{Q})$$

$$\Rightarrow \dot{\boldsymbol{Q}} = \frac{\dot{\gamma}(t)}{2}\left[\begin{bmatrix} Q_{12} & -Q_{11} \\ -Q_{11} & -Q_{12} \end{bmatrix} - \begin{bmatrix} -Q_{12} & Q_{11} \\ Q_{11} & Q_{12} \end{bmatrix}\right] + a\frac{\dot{\gamma}(t)}{2}\left[\begin{bmatrix} Q_{12} & -Q_{11} \\ Q_{11} & Q_{12} \end{bmatrix} + \begin{bmatrix} Q_{12} & Q_{11} \\ -Q_{11} & Q_{12} \end{bmatrix}\right]$$

$$+a\frac{\dot{\gamma}(t)}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 2a\dot{\gamma}(t)Q_{12}\left(\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) - 6D_r^0 F(\boldsymbol{Q})$$

$$\Rightarrow \dot{\boldsymbol{Q}} = \frac{\dot{\gamma}(t)}{2}\begin{bmatrix} 2Q_{12} & -2Q_{11} \\ -2Q_{11} & -2Q_{12} \end{bmatrix} + a\frac{\dot{\gamma}(t)}{2}\begin{bmatrix} 2Q_{12} & 0 \\ 0 & 2Q_{12} \end{bmatrix} + a\frac{\dot{\gamma}(t)}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$-2a\dot{\gamma}(t)Q_{12}\left(\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) - 6D_r^0 F(\boldsymbol{Q})$$

$$\Rightarrow \dot{\boldsymbol{Q}} = \dot{\gamma}(t)\begin{bmatrix} Q_{12} & -Q_{11} \\ -Q_{11} & -Q_{12} \end{bmatrix} + a\dot{\gamma}(t)\begin{bmatrix} Q_{12} & 0 \\ 0 & Q_{12} \end{bmatrix} + a\frac{\dot{\gamma}(t)}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$-2a\dot{\gamma}(t)Q_{12}\left(\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$-6D_r^0\left\{\begin{aligned} &\left(1-\frac{N}{2}\right)\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} - N\begin{bmatrix} \left(Q_{11}^2+Q_{12}^2\right) & 0 \\ 0 & \left(Q_{11}^2+Q_{12}^2\right) \end{bmatrix} \\ &\qquad + 2N\left(Q_{11}^2+Q_{12}^2\right)\begin{bmatrix} Q_{11}+\frac{1}{2} & Q_{12} \\ Q_{12} & -Q_{11}+\frac{1}{2} \end{bmatrix} \end{aligned}\right\}$$

$$\Rightarrow \dot{\boldsymbol{Q}} = -6D_r^0 \begin{pmatrix} \left(1-\frac{N}{2}\right)\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} - N\begin{bmatrix} \left(Q_{11}^2+Q_{12}^2\right) & 0 \\ 0 & \left(Q_{11}^2+Q_{12}^2\right) \end{bmatrix} \\ +2N\left(Q_{11}^2+Q_{12}^2\right)\begin{bmatrix} Q_{11}+\frac{1}{2} & Q_{12} \\ Q_{12} & -Q_{11}+\frac{1}{2} \end{bmatrix} \end{pmatrix} \tag{23}$$

$$+\left\{ \begin{array}{l} \begin{bmatrix} Q_{12} & -Q_{11} \\ -Q_{11} & -Q_{12} \end{bmatrix} + a\begin{bmatrix} Q_{12} & 0 \\ 0 & Q_{12} \end{bmatrix} + \frac{a}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ -2aQ_{12}\left(\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & -Q_{11} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \end{array} \right\} \dot{\gamma}(t)$$

The system's components are as follows:

$$\dot{Q}_{11} = -6D_r^0 Q_{11}\left(1-\frac{N}{2}+2N\left(Q_{11}^2+Q_{12}^2\right)\right) + Q_{12}\left(1-2aQ_{11}\right)\dot{\gamma}(t)$$

$$\dot{Q}_{12} = -6D_r^0 Q_{12}\left(1-\frac{N}{2}+2N\left(Q_{11}^2+Q_{12}^2\right)\right) + \left(-Q_{11}+\frac{a}{2}-2aQ_{12}^2\right)\dot{\gamma}(t)$$

Regard $\dot{\gamma}(t)$ as a control input and consider the problem (19) with initial state Q_0 and final state Q_1. The system is controllable if for any choices of Q_0 and Q_1 there exists a control $\dot{\gamma}(t)$ on the interval $(0,T)$.

Using the nematic relaxation time scale $\frac{1}{D_r^0}$, the flow field and orientation dynamics of (19) can be non-dimensionalized. The key dimensionless parameters are then the Peclet number $Pe(t) = \frac{\dot{\gamma}}{D_r^0}$ (the shear rate normalized with respect to nematic relaxation rate) and the dimensionless concentration parameter N.

$$\Rightarrow \dot{Q}_{11} = D_r^0\left[-6Q_{11}\left(1-\frac{N}{2}+2N\left(\left(Q_{11}\right)^2+\left(Q_{12}\right)^2\right)\right) + Q_{12}\left(1-2aQ_{11}\right)Pe(t)\right]$$

$$\Rightarrow \dot{Q}_{12} = D_r^0\left[-6Q_{12}\left(1-\frac{N}{2}+2N\left(\left(Q_{11}\right)^2+\left(Q_{12}\right)^2\right)\right)+\left(-Q_{11}+\frac{a}{2}-2a\left(Q_{12}\right)^2\right)Pe(t)\right]$$

Applying $\bar{t} = tD_r^0$, the equations become dimensionless and the nematodynamic model (23) can be expressed as:

$$\begin{aligned}
&\Rightarrow \dot{Q}_{11} = -6Q_{11}\left(1-\frac{N}{2}+2N\left(\left(Q_{11}\right)^2+\left(Q_{12}\right)^2\right)\right)+Q_{12}\left(1-2aQ_{11}\right)Pe(t) \\
&\Rightarrow \dot{Q}_{12} = -6Q_{12}\left(1-\frac{N}{2}+2N\left(\left(Q_{11}\right)^2+\left(Q_{12}\right)^2\right)\right)+\left(-Q_{11}+\frac{a}{2}-2a\left(Q_{12}\right)^2\right)Pe(t)
\end{aligned} \tag{24}$$

For mathematical convenience, the following notion is introduced:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} Q_{11} \\ Q_{12} \end{bmatrix}$$

Then system (24) can be written as:

$$\frac{d\vec{x}}{dt} = \vec{f}\left(\vec{x}\right)+\vec{g}\left(\vec{x}\right)u$$

Where $\vec{f}\left(\vec{x}\right) = \begin{bmatrix} -6x_1\left(1-\frac{N}{2}+2N\left(x_1^2+x_2^2\right)\right) \\ -6x_2\left(1-\frac{N}{2}+2N\left(x_1^2+x_2^2\right)\right) \end{bmatrix}$, $\vec{g}\left(\vec{x}\right) = \begin{bmatrix} x_2\left(1-2ax_1\right) \\ -x_1+\frac{a}{2}-2ax_2^2 \end{bmatrix}$, $u = Pe(t)$

$$\Rightarrow \begin{bmatrix} Q_{11} \\ Q_{12} \end{bmatrix} = \begin{bmatrix} -6x_1\left(1-\frac{N}{2}+2N\left(x_1^2+x_2^2\right)\right) \\ -6x_2\left(1-\frac{N}{2}+2N\left(x_1^2+x_2^2\right)\right) \end{bmatrix} + \begin{bmatrix} x_2\left(1-2ax_1\right) \\ -x_1+\frac{a}{2}-2ax_2^2 \end{bmatrix} Pe(t)$$

Compute the following Lie Bracket where $\left[\vec{f},\vec{g}\right] = \nabla\vec{g}\cdot\vec{f} - \nabla\vec{f}\cdot\vec{g}$.

$$\Rightarrow\left[\vec{f},\vec{g}\right]=\begin{bmatrix}-2ax_2 & 1-2ax_1\\ -1 & -4ax_2\end{bmatrix}\cdot\begin{bmatrix}-6x_1\left(1-\frac{N}{2}+2N\left(x_1^2+x_2^2\right)\right)\\ -6x_2\left(1-\frac{N}{2}+2N\left(x_1^2+x_2^2\right)\right)\end{bmatrix}$$

$$-\begin{bmatrix}-6\left(1-\frac{N}{2}+2N\left(3x_1^2+x_2^2\right)\right) & -24Nx_1x_2\\ -24Nx_1x_2 & -6\left(1-\frac{N}{2}+2N\left(x_1^2+3x_2^2\right)\right)\end{bmatrix}\cdot\begin{bmatrix}x_2\left(1-2ax_1\right)\\ -x_1+\frac{a}{2}-2ax_2^2\end{bmatrix}$$

$$\Rightarrow\left[\vec{f},\vec{g}\right]=\begin{bmatrix}-6x_2\left(1-\frac{N}{2}+2N\left(x_1^2+x_2^2\right)\right)\left[1-4ax_1\right]\\ 6\left(1-\frac{N}{2}+2N\left(x_1^2+x_2^2\right)\right)\left[x_1+4ax_2^2\right]\end{bmatrix}$$

$$-\begin{bmatrix}-6x_2\left(1-\frac{N}{2}+2N\left(3x_1^2+x_2^2\right)\right)\left(1-2ax_1\right)+24Nx_1x_2\left(x_1-\frac{a}{2}+2ax_2^2\right)\\ -24Nx_1x_2^2\left(1-2ax_1\right)+6\left(1-\frac{N}{2}+2N\left(x_1^2+3x_2^2\right)\right)\left(x_1-\frac{a}{2}+2ax_2^2\right)\end{bmatrix}$$

$$\Rightarrow\left[\vec{f},\vec{g}\right]=\begin{bmatrix}12ax_1x_2\left[1+\frac{N}{2}-2N\left(x_1^2+x_2^2\right)\right]\\ 3a\left(4x_2^2+1+4N\left[-\frac{1}{8}-2x_2^2\left(x_1^2+x_2^2\right)+\frac{1}{2}\left(x_1^2+2x_2^2\right)\right]\right)\end{bmatrix}\tag{25}$$

Construct a matrix with the vectors from $\vec{g}\left(\vec{x}\right)$ and (25) to compute the determinant: $\det\left[\vec{g},\left[\vec{f},\vec{g}\right]\right]$.

$$\Rightarrow\det\begin{bmatrix}x_2\left(1-2ax_1\right) & 12ax_1x_2\left[1+\frac{N}{2}-2N\left(x_1^2+x_2^2\right)\right]\\ -x_1+\frac{a}{2}-2ax_2^2 & 3a\left(4x_2^2+1+4N\left[-\frac{1}{8}-2x_2^2\left(x_1^2+x_2^2\right)+\frac{1}{2}\left(x_1^2+2x_2^2\right)\right]\right)\end{bmatrix}$$

$$=x_2\left(1-2ax_1\right)3a\left(4x_2^2+1+4N\left[-\frac{1}{8}-2x_2^2\left(x_1^2+x_2^2\right)+\frac{1}{2}\left(x_1^2+2x_2^2\right)\right]\right)$$

$$-12ax_1x_2\left[1+\frac{N}{2}-2N\left(x_1^2+x_2^2\right)\right]\left(-x_1+\frac{a}{2}-2ax_2^2\right)$$

$$= 3ax_2 \left\{ \begin{array}{l} (1-2ax_1)\left(4x_2^2+1\right)+4N(1-2ax_1)\left[-\frac{1}{8}-2x_2^2\left(x_1^2+x_2^2\right)+\frac{1}{2}\left(x_1^2+2x_2^2\right)\right] \\ \qquad -4x_1\left(-x_1+\frac{a}{2}-2ax_2^2\right)+\left[-2Nx_1+8Nx_1\left(x_1^2+x_2^2\right)\right]\left(-x_1+\frac{a}{2}-2ax_2^2\right) \end{array} \right\}$$

$$= 3ax_2 \left\{ \begin{array}{l} (1-2ax_1)\left(4x_2^2+1\right)-4x_1\left(-x_1+\frac{a}{2}-2ax_2^2\right) \\ +2N\left\{ \begin{array}{l} -\frac{1}{4}-4x_2^2\left(x_1^2+x_2^2\right)+\left(x_1^2+2x_2^2\right)-2ax_1\left[-\frac{1}{4}-4x_2^2\left(x_1^2+x_2^2\right)+\left(x_1^2+2x_2^2\right)\right] \\ \qquad +x_1\left[-1+4\left(x_1^2+x_2^2\right)\right]\left(-x_1+\frac{a}{2}-2ax_2^2\right) \end{array} \right\} \end{array} \right\}$$

$$= 3ax_2 \left\{ \begin{array}{l} 4\left(x_1^2+x_2^2\right)+1-4ax_1 \\ +2N\left\{ \begin{array}{l} -\frac{1}{4}-4x_2^2\left(x_1^2+x_2^2\right)-4x_1^2\left(x_1^2+x_2^2\right)+\left(x_1^2+x_1^2+2x_2^2\right) \\ -2ax_1\left[\frac{1}{4}-\frac{1}{4}-4x_2^2\left(x_1^2+x_2^2\right)+\left(x_1^2+2x_2^2\right)-x_2^2-\left(x_1^2+x_2^2\right)+4x_2^2\left(x_1^2+x_2^2\right)\right] \end{array} \right\} \end{array} \right\}$$

$$= 3ax_2 \left\{ \begin{array}{l} 4\left(x_1^2+x_2^2\right)+1-4ax_1 \\ \qquad +2N\left[-\frac{1}{4}-4\left[x_2^2+x_1^2\right]\left(x_1^2+x_2^2\right)+2\left(x_1^2+x_2^2\right)\right] \end{array} \right\}$$

$$= 3ax_2\left[N\left[-\frac{1}{2}-8\left(x_1^2+x_2^2\right)^2+4\left(x_1^2+x_2^2\right)\right]+4\left(x_1^2+x_2^2\right)+1-4ax_1\right]$$

$$\Rightarrow \det\left[\vec{g},\left[\vec{f},\vec{g}\right]\right] = 3ax_2\left[4\left(x_1^2+x_2^2\right)\left(-2N\left(x_1^2+x_2^2\right)+N+1\right)-\frac{N}{2}+1-4ax_1\right]$$

The determinant equals zero where $x_2 \neq 0$ and $4\left(x_1^2+x_2^2\right)\left(-2N\left(x_1^2+x_2^2\right)+N+1\right)-\frac{N}{2}+1-4ax_1 \neq 0$. It can be concluded that the controllability of the sub system from the Doi model under shear flow satisfies Definition 3 and thus by Theorem 1, it is weakly controllable when $x_2 \neq 0$ (or $Q_{12} \neq 0$) and $4\left(x_1^2+x_2^2\right)\left(-2N\left(x_1^2+x_2^2\right)+N+1\right)-\frac{N}{2}+1-4ax_1 \neq 0$

(or $4\left(Q_{11}^2+Q_{12}^2\right)\left(-2N\left(Q_{11}^2+Q_{12}^2\right)+N+1\right)-\frac{N}{2}+1-4aQ_{11}\neq 0$). This result can be summarized geometrically:

Given $R^2=\left\{\left(x_1,x_2\right)\mid x_1,x_2\in R\right\}$, there exists two defined surfaces:

$$S_1=\left\{\left(x_1,x_2\right)\mid x_2=0\right\}$$
$$S_2=\left\{\left(x_1,x_2\right)\mid 4\left(x_1^2+x_2^2\right)\left(-2N\left(x_1^2+x_2^2\right)+N+1\right)-\frac{N}{2}+1-4ax_1=0\right\}$$

such that the system is weakly controllable at all points in $R^3\setminus\left\{S_1\cup S_2\right\}$.

C. SUMMARY OF THE DOI MODEL RESULTS

The sub system from the Doi model under both the homogeneous extensional flow and shear flow showed to be weakly controllable via the Lie bracket method.

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V. SUMMARY OF MODELS IN TABLE

The following table depicts a snapshot of the results found in this thesis.

MODEL	**Homogeneous Extensional Flow**	**Homogeneous Shear Flow**
Phan-Thien-Tanner	weakly controllable when $T_{11} \neq \pm T_{22}$ and $T_{11} \neq -\mu$ and $T_{12} \neq 0$	weakly controllable when $T_{22} \neq 0$ and $T_{12} \neq 0$ and $\kappa T_{11} T_{22} \left[\mu + T_{22} \right] + 2 T_{12}^2 \left[\lambda + \kappa T_{11} \right] = \frac{\mu \lambda}{\kappa}$
Johnson-Segalman	weakly controllable when $T_{11} \neq T_{22}$	undetermined with the chosen vectors
Doi	weakly controllable when $Q_{11} \neq 0$ and $Q_{12} \neq 0$	weakly controllable when $Q_{12} \neq 0$ and $\left(Q_{11}^2 + Q_{12}^2 \right) \left(-2N \left(Q_{11}^2 + Q_{12}^2 \right) + N + 1 \right) - a Q_{11} \neq \frac{N-2}{8}$

Table 1. Constitutive Model Overview

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VI. CONCLUDING REMARKS AND FUTURE WORK

The work covered in this thesis investigated the controllability of three popular constitutive models under homogeneous extensional and shear flows via the Lie bracket method. By applying tools from the geometric control theory, submanifolds on which each system or its subsystem is weakly controllable were explicitly identified. Characterizing a system by means of this method is a fairly new concept and it has applicability to a wider variety of models.

Potential extensions of this paper include the investigation of controllability in three-dimensional systems, though it definitely becomes quite challenging. Note that this thesis only gave weakly controllable set. Identifying reachable set for each model under extensional flow should be a future work. Another possibility for future work is the exploration of controllability of the state equation under nonhomogeneous flows. This requires analysis of PDE systems and it is undoubtedly more complicated.

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